MATHEMATICS

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THE EDITOR'S PAGE

Announcement

Commencing with Volume 35, the MATHEMATICS MAGAZINE will become an official publication of The Mathematical Association of America. It will be owned and supported by the Association. Friends and readers of the MAGAZINE will be pleased to know that such a strong and prominent professional organization saw fit to guarantee the future of the MAGAZINE as a journal devoted to collegiate mathematics.

The Association is in the process of taking over the management of the MAGAZINE at the present time. Please note that all business correspondence, subscription requests, and changes of address should be sent to H. M. Gehman, Executive Director of the Association at the University of Buffalo. Check the correct mailing address on the inside front cover. Articles and editorial correspondence should still be sent to the editorial office at Los Angeles City College.

It is anticipated that the editorial policies and format of the MAGAZINE will not be substantially changed. The MAGAZINE will continue to aim for a level somewhere between the AMERICAN MATHEMATICAL MONTHLY and the MATHEMATICS TEACHER. A wide variety of mathematical interests will be served.

Sponsoring Subscribers

In recent years a number of mathematicians have contributed to the financial support of the MATHEMATICS MAGAZINE by purchasing sponsoring subscriptions. These have aided materially in maintaining the MAGAZINE. Now that the Association is assuming the full financial support of the MAGAZINE the sponsoring subscriptions are being discontinued. No further sponsoring subscriptions are being accepted. The names of our current sponsors will continue to appear in the title page until the last issue of this volume.

The mathematical profession owes a debt of sincere gratitude to the sponsors who have supported the MAGAZINE so faithfully and generously throughout the years. The editors wish to add their personal word of thanks for this loyal and encouraging support.

R. E. H.

RESISTANCE CIRCUITS AND THINGS SYNTHESIZED
BY NUMBER THEORY

Arne Benson

Many youthful imaginations have been fired by the similarity between the mathematical laws of combination of such diverse physical phenomena as resistances in parallel, capacitances in series, inductances, lenses and springs. A study of the problem might reveal other measurable phenomena (say r_1 and r_2) which combine to produce a total effect, R according to the same law,

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} \ .$$

A tub will fill with bathwater in time R if the hot and cold water spigots would separately fill the tub in times r_1 and r_2 respectively. Soap bubbles of radii r_1 and r_2 coalesce to form a common interface with radius R according to the same law. The engineer can readily produce other examples — some exquisitely pretty, others fair but mundane — from his bag of tricks and handy formulas.

With the observation that there exists identity of form in such diverse phenomena, one man might be led into deep philosophical speculations upon the nature of order in the universe, while his more practical brother might shrug it off as "interesting but who needs it?" An extremist of the former persuasion might pursue the philosophical vision at all costs while his opposite can be imagined campaigning vigorously against such impractical boondoggles. Depending upon ones system of values, speculations such as these will rate a position somewhere along the real number axis from minus to plus infinity.

The amazing collection of mathematical topics which we call the theory of numbers and the British know as the Higher Arithmetic, in similar vein, qualifies to be rated somewhere on the closed interval from the ridiculous to the sublime, its exact position again depending upon the individual system of values. There is no doubt, however, that the subject has intrigued the amateur and challenged the keenest resources of the greatest professionals since Thales. The source of difficulty in this subject lies in the restriction to whole numbers in the problems it considers. But this appears exotic to us who work with the results of the physical sciences and adopt, or inherit, the assumption that physical phenomena are continuous and their measure capable of occupying any point at all along the real line or in manifolds of higher dimensionality. Once in a great while, a genuine number theory problem finds physical application and more rarely,

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may bring direct economic returns. Digital computers illustrate this point admirably, and some mechanical engineers are aware of deep number-theoretical implications in the synthesis of trains of gears. Toothed wheels, as our predecessors called them, are simply not made with a continuous range of tooth numbers.

It may not, then, be completely amiss nor wholly a boondoggle to consider the law

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

from a number theoretic viewpoint. At the very least, it will be an interesting mathematical exercise and some may even find a peculiar beauty in the derivations, and possibly a mild astonishment at the historic connections which unfold. The methods are quite general and adaptable to the solution of many second degree Diophantine (solutions in integers only) equations, consequently may find place in the engineer's bag of tricks. At the very most, some modest application to the class of phenomena mentioned earlier may occur to reward the reader.

To make the problem specific, consider how to select two resistors of integer values r_1 and r_2 which when connected in parallel will yield a desired circuit resistance R, also a whole number. More concisely,

Given R, an integer > 0, solve for positive integers r_1 and r_2, in

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} .$$

The analysis proceeds as follows. Solve for R to get the familiar

(1) $$R = \frac{r_1 r_2}{r_1 + r_2} .$$

At this point, choose to eliminate the bilinear term, $r_1 r_2$ by making the substitution, $r_2 = r_1 + V$, this being a fair example of the remarkable mathematical foresight displayed in textbooks and upon learned blackboards and which is many times the result of having previously — and quite honorably so — stumbled up and down the alternate avenues of reasoning. The substitution yields

$$R = \frac{r_1(r_1 + V)}{r_1 + (r_1 + V)}$$

which is easily transformed into aquadratic in r (dropping the tedious subscript with the mental note to pick it up later)

(2) $$r^2 + (V - 2R)r - RV = 0 .$$

That keystone of high school Algebra II, the quadratic formula,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

gives the solution to the equation,

$$Ax^2 + Bx + C = 0$$

but does not give any direct advice on how to find the integer valued roots required for equation (2).

But notice, as did Fermat, Euler and others in similar circumstances, that in order for the roots to be integral, the discriminant, $B^2 - 4AC$ must necessarily be the square of an integer, or how otherwise could its square root be an integer? Write this most crucial condition as, $Y^2 = B^2 - 4AC$ and substitute the coefficients of (2) into the quadratic formula to get

$$r = \frac{-(V - 2R) \pm \sqrt{(V - 2R)^2 + 4RV}}{2}$$

which cleans up to

$$r = \frac{2R - V \pm \sqrt{V^2 + 4R^2}}{2}$$

or further, if that very crucial observation about the discriminant is plugged in,

(3) $$r = \frac{2R - V \pm Y}{2}$$

wherein

(4) $$Y^2 = V^2 + 4R^2.$$

At first glance, this may not appear to offer any advantage over the original equation, but if with equal prescience, the transformation, $X = 2R$, is made in equation (4), the neat Pythagorean formula results,

(5) $$Y^2 = V^2 + X^2.$$

Now, here is an equation with a history and a tremendous significance. Its multitudinous threads run through every branch of mathematics and most of physics, classical and modern. One of the tasks Euclid set himself was to find its integer solutions, to give in effect right triangles with integer valued sides. He found special solutions, as did Pythagoras and Plato, but did probably not know the general solution,

$$Y = \alpha^2 + \beta^2$$
$$V = \alpha^2 - \beta^2$$
$$X = 2\alpha\beta$$

where α and β are integers, $\alpha > \beta$, and in the case where primitive (the smallest of an infinite set of similar triangles) solutions are specified,

$\propto$ and β relatively prime and not both odd. This noteworthy result was given later, in the 4 th century A. D. by Diophantus, and continues to be a source of thematic material from which endless variations are expounded to this day.

Taking advantage of the gratuitous solution of equation (5) and observing that

$$X = 2R = 2\propto\beta$$

it is clear that, if use is to be made of Diophantus' gift, R must equal $\propto\beta$, the product of the parameters in his ancient solution.

Return to equation (3) which was given as

(3)
$$r = \frac{2R - V \pm Y}{2}$$

the variables, V and Y, are known in terms of the parameters, $\propto$ and β, which in turn are complimentary divisors of R. Therefore, substitute

$$Y = \propto^2 + \beta^2 \quad \text{and} \quad V = \propto^2 - \beta^2$$

into equation (3) to give

$$r = \frac{2R - (\propto^2 - \beta^2) \pm (\propto^2 + \beta^2)}{2}$$

which results in

(7)
$$r = R + \beta^2 \quad \text{and} \quad r = R - \propto^2$$

depending upon choice of the ambiguous sign.

Picking up the subscripts again and recalling that it was decided to let

$$r = r_1 \quad \text{and} \quad r_2 = r_1 + V$$

and moreover, substituting the parametric value of $V = (\propto^2 - \beta^2)$ into the definition of r_2 yields two solution sets

(8)
$$r_1 = R + \beta^2$$
$$r_2 = R + \propto^2$$

and

(9)
$$r_1 = R - \propto^2$$
$$r_2 = R - \beta^2$$

depending upon which root (7) of the quadratic is employed.

It is clear, as mathematicians are fond of remarking, that only set (8) represents a physically acceptable solution to the problem of resistances in parallel as originally stated, since set (9) will always result in an impossibly negative resistance. The solution to the problem may now be

reduced to the recipe

To find two integral valued resistors r_1 and r_2 whose parallel circuit value is a specified whole number R, factor R into $\propto \cdot \beta = R$, then for each pair of divisors $\propto$ and β

$$r_1 = R + \beta^2$$

$$r_2 = R + \propto^2 .$$

The slightly more general case, which does not discriminate against negative values and applies to other physical phenomena of similar type, deserves to be stated.

To solve $R = r_1 r_2/(r_1 + r_2)$ in integers factor R into $R = \propto \cdot \beta$, then

$$r_1 = R \pm \beta^2$$

$$r_2 = R \pm \propto^2$$

(signs taken alike).

A small detail needs to be cleaned up before a legitimate claim to generality can be made. As everyone who has ever soldered in a resistor knows, but which might easily escape the erudite mathematician, there exists one other solution; namely the one which corresponds to the factorization, $\propto = \beta = \sqrt{R}$. In this unique case, $r_1 = r_2 = 2R$ for resistors, but vanishes for the negative choice of sign in the general solution.

Many recipes for r_1 and r_2 where R is of special form — say, R even, or divisible by ten, or something of that sort — may be concocted for the amusement of friends and junior associates. It is always certain that at least one solution exists; the one corresponding to the factorization $R = \propto \cdot \beta = 1 \cdot R$, and it may also be observed that an interchange of $\propto$ and β does not yield a different solution since the original equation was symmetrical in the variables.

The question naturally arises concerning the exact number of different solutions for some given R. Denote this number by $N(R)$ and recall that each factorization, disregarding permutations, yielded a solution. Let $d(R)$ denote the number of divisors of R including R and unity, then clearly

and
$$N(R) = \tfrac{1}{2} d(R) + 1 \quad \text{if } R \text{ is not a square}$$
$$N(R) = \tfrac{1}{2} d(R) + \tfrac{1}{2} \quad \text{if } R \text{ is a square} .$$

An explicit expression for $d(R)$ is given by the Theory of Numbers. Express R in its prime factor form (which is unique),

$$R = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} ,$$
then
$$N(R) = (k_1 + 1)(k_2 + 1) \cdots (k_n + 1) .$$

Consequently, the number of different ways, $N(R)$, in which two integer

valued resistors can be paralleled to give an integral circuit value R is

$$(10) \qquad\qquad N(R) = \tfrac{1}{2} \prod_{t=1}^{n} (k_t + 1) + W$$

where W is $\tfrac{1}{2}$ or unity according as R is or is not a square and

$$R = \prod_{i=1}^{n} p_i^{k_i}$$

is the prime factorization of the desired circuit value, R.

For the sake of a numerical specimen, consider, $R = 120$. Its prime decomposition is $120 = 2^3 \cdot 3 \cdot 5$, which gives the exponents $k_1 = 3$, $k_2 = 1$, $k_3 = 1$ and, upon invoking the powerful looking equation (10), yields the number of solutions

$$N(120) = \tfrac{1}{2}(3+1)(1+1)(1+1) + 1 = 9 \ .$$

These correspond to the factorizations

$(\alpha \cdot \beta) = (1 \cdot 120), \ (2 \cdot 60), \ (3 \cdot 40), \ (4 \cdot 30), \ (5 \cdot 24), \ (6 \cdot 20), \ (8 \cdot 15), \ (10 \cdot 12),$
$\qquad (\sqrt{120} \cdot \sqrt{120}) \ .$

and in the same order are the solutions,

$(r_1, r_2) \ = \ (121, 14520), \ \ (124, 3720), \ \ (129, 1720), \ \ (136, 1020), \ \ (145, 696),$
$\qquad\qquad (156, 520), \ (184, 345), \ (220, 264), \ (240, \ 240) \ .$

The analysis stops at this point, although the topic is by no means exhausted. But of what use is it? Perhaps only an excercise, certainly interesting in its historical connections; and maybe it has esthetic value for some. On the other hand, maybe it can build a digital computer or figure a horse race — horses, like computers, are generally discrete entities and their tangential velocities, reciprocals of track time. Perhaps with it, one might devise a better "decade" box or optical system. Finally, perhaps it should be consigned to the drain with both spigots running. But, if this is the verdict, who can say it's an eternally just one.

14 Brookline Street
Nashua, New Hampshire

THE STAUDT-CLAUSEN THEOREM

L. Carlitz

1. The Bernoulli numbers may be defined by means of

$$(1.1) \qquad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \, ,$$

which is equivalent to

$$(B+1)^n - B^n \quad (n > 1) \, ,$$

where after expansion B^r is replaced by B_r. More generally the Bernoulli polynomial is defined by means of

$$(1.2) \qquad \frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \, ;$$

This implies $B_n(0) = B_n$,

$$(1.3) \qquad B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r} x^r \, ,$$

$$(1.4) \quad B_n(x+1) - B_n(x) = n x^{n-1} \, , \quad B_1 = -\tfrac{1}{2} \, , \quad B_{2n+1} = 0 \quad (n \geq 1) \, .$$

The numbers B_{2n} are rational; the fractional part of B_{2n} is determined by

$$(1.5) \qquad B_{2n} = G_{2n} - \sum_{p-1 \mid 2n} \frac{1}{p} \quad (n \geq 1) \, ,$$

where G_{2n} is an integer and the summation is over all primes p such that $p-1 \mid 2n$. For example we have

$$B_2 = 1 - \frac{1}{2} - \frac{1}{3} \, , \quad B_4 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \, , \quad B_6 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} \, ,$$

$$B_8 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \, , \quad B_{10} = 1 - \frac{1}{2} - \frac{1}{3} - 11 \, .$$

We remark that (1.5) implies

$$(1.6) \qquad p B_{2n} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid 2n) \\ 0 \pmod{p} & (p-1 \nmid 2n) \end{cases} ;$$

conversely (1.6) implies (1.5).

The formula (1.5) is known as the Staudt-Clausen theorem. The object of the present paper is to give several proofs of (1.3) and some related results and also to derive a rather general theorem of which (1.5) is a special case. For references to the Staudt-Clausen theorem we mention in particular [2], [5], [6], [8], [9], [10], [11], [12], [13].

2. We have

$$t = \log[1+(e^t-1)] = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r}(e^t-1)^r ,$$

$$\frac{t}{e^t-1} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1}(e^t-1)^r .$$

Since

$$(e^t-1)^r = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}e^{st} = \sum_{n=r}^{\infty}\frac{t^n}{n}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^n ,$$

it follows that

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{r=0}^{n}\frac{1}{r+1}\sum_{s=0}^{r}(-1)^s\binom{r}{s}s^n .$$

Comparison with (1.1) yields

(2.1)
$$B_n = \sum_{r=0}^{n}\frac{1}{r+1}\sum_{s=0}^{r}(-1)^s\binom{r}{s}s^n .$$

This formula can also be obtained without the use of infinite series as follows. We have from (1.4)

$$\sum_{j=0}^{k-1} j^n = \frac{1}{n+1}(B_{n+1}(k)-B_{n+1}) .$$

On the other hand, by finite differences,

(2.2)
$$x^n = \sum_{r=0}^{n}\binom{x}{r}A_{nr} ,$$

where

$$A_{nr} = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^n .$$

Then

$$\sum_{j=0}^{k-1} j^n = \sum_{r=0}^{n} \binom{k}{r+1} A_{nr} ,$$

so that

$$(2.3) \qquad \sum_{r=0}^{n} \binom{k}{r+1} A_{nr} = \frac{1}{n+1}(B_{n+1}(k) - B_{n+1}) .$$

This is true for all $k = 1, 2, 3, \cdots$ and is therefore an identity in k. If we now divide both sides of (2.3) by k and then put $k = 0$ we get (2.1) at once.

By means of (2.1) we can give a simple proof of (1.5). We require the fact that A_{nr} is divisible by $r!$; if we rewrite (2.2) in the form

$$x^n = \sum_{r=0}^{n} \frac{1}{r!} A_{nr} x(x-1) \cdots (x-r+1) ,$$

it is clear that the coefficient $A_{nr}/r!$ is integral. Then by (2.1)

$$(2.4) \qquad B_{2n} = \sum_{r=0}^{n} (-1)^r \frac{r!}{r+1} \frac{1}{r!} A_{2n,r} .$$

Now if $r+1$ is composite and ≥ 6, put $r+1 = ab$, where $a \geq 2$, $b \geq 2$. It follows at once that $r!$ is divisible by ab so that $r!/(r+1)$ is integral. Next if $r+1 = 4$, we have

$$A_{2n,3} = 3^{2n} - 3 \cdot 2^{2n} + 3 \cdot 1^{2n} \equiv 1 + 3 \equiv 0 \quad (\bmod\ 4) .$$

We note for a later purpose that

$$A_{2n+1,3} = 3^{2n+1} - 3 \cdot 2^{2n+1} + 3 \cdot 1^{2n+1} \equiv 3 + 3 \equiv 2 \quad (\bmod\ 4)$$

for $n > 0$, that is

$$(2.5) \qquad A_{2n+1,3} \equiv 2 \quad (\bmod\ 4) \qquad (n \geq 1) .$$

Finally let $r+1 = p$, a prime. Then

$$A_{2n,p-1} = \sum_{s=0}^{p-1} (-1)^s \frac{p-1}{s} s^n \equiv \sum_{s=0}^{p-1} s^n \quad (\bmod\ p) ,$$

so that

$$A_{2n,p-1} \equiv \begin{cases} -1 & (\bmod\ p) \qquad (p-1 \mid 2n) \\ 0 & (\bmod\ p) \qquad (p-1 \nmid 2n) . \end{cases}$$

We may now state

Theorem 1. *If n is even and ≥ 2 then*

$$B_n = G_n - \sum_{p-1 \mid n} \frac{1}{p} \, ,$$

where G_n is integral and the summation is over all primes p such that $p-1 \mid n$.

The above proof is due to Lucas [8].

3. Let h, k be relatively prime integers and put

(3.1) $$b_n = b_n(h, k) = k^n B_n(h/k) \, .$$

We shall now show how to obtain a theorem analogous to (1.5) for the numbers b_n. In the first place we have from (1.3) and (2.1)

$$B_n(x) = \sum_{r=0}^{n} \frac{1}{r+1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (x+s)^n$$

and therefore

(3.2) $$b_n(h, k) = \sum_{r=0}^{n} \frac{1}{r+1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (h+ks)^n \, .$$

As above

$$A_{nr} = \sum_{s=0}^{r} (-1)^s \binom{r}{s} (h+ks)^n$$

is divisible by $r!$, so that those terms in the right member of (3.2) for which $r+1$ is composite and ≥ 6 are integral. For $r+1 = 4$ we have several subcases.

(i) If k is even it is clear that

$$\Delta_3 \equiv 0 \quad (\text{mod } 4) \, .$$

(ii) If k is odd while n is even we get

$$\Delta_3 \equiv (h+k)^n - (h-k)^n \equiv 0 \quad (\text{mod } 4) \, .$$

(iii) If k and n are both odd and $n > 1$ then

$$2\Delta_1 + \Delta_3 \equiv -\sum_{s=0}^{3} (h+sk)^n \equiv -\sum_{s=0}^{3} s^n \equiv 0 \quad (\text{mod } 4) \, ,$$

while for $n = 1$ we get

$$2\Delta_1 + \Delta_3 \equiv 0 + 1 + 2 + 3 \equiv 2 \quad (\text{mod } 4) \, .$$

For $r+1 = p$, where $p \mid k$, we have

$$\Delta_{p-1} \equiv \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} h^n \equiv 0 \quad (\text{mod } p) ,$$

while if $p \nmid k$ we get

$$\Delta_{p-1} \equiv \sum_{s=0}^{p-1} (h+sk)^n \equiv \sum_{s=0}^{p-1} s^n \equiv \begin{cases} -1 & (\text{mod } p) & (p-1 \mid n) \\ 0 & (\text{mod } p) & (p-1 \nmid n) . \end{cases}$$

We may now state the following result which is due to Vandiver [14].

Theorem 2. *If h and k are relatively prime and $b_n = b_n(h, k)$ is defined by* (3.1), *then if n is even*

$$b_n(h, k) = G_n - \sum \frac{1}{p} ,$$

where G_n is an integer and the summation is over all primes p such that $p-1 \mid n$ but $p \nmid k$. If n is odd, $b_n(h, k)$ is an integer, except for $n = 1$, k odd, in which case

$$b_1(h, k) = G_1 + \tfrac{1}{2} .$$

For $h = 0$, $k = 1$, Theorem 2 reduces to the Staudt-Clausen theorem. It is of interest to observe that if k is divisible by all primes p such that $p-1 \mid n$ then it follows at once from Theorem 2 that $b_n(h, k)$ is integral.

4. Put [10, p. 28]

(4.1) $$D_n = 2^n B_n(\tfrac{1}{2}) .$$

It follows easily from (1.2) that

$$\frac{t}{\sinh t} = \sum_{n=0}^{\infty} D_{2n} \frac{t^{2n}}{(2n)!} ;$$

also $D_{2n+1} = 0$. From the formula

$$B_n(x) + B_n(x + \tfrac{1}{2}) = 2^{1-n} B_n(2x)$$

we get

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n ,$$

so that

(4.2) $$D_n = (2 - 2^n)B_n .$$

If D_n is written as the quotient of two relatively prime integers, it is

clear that the denominator is odd. Moreover, using (1.6), we get for odd p

$$pD_{2n} = (2 - 2^{2n})pB_{2n} \equiv 0 \pmod{p}$$

if $p - 1 \nmid 2n$, so that D_{2n} is integral (mod p) for such p. On the other hand, if $p - 1 \mid 2n$, then

$$pD_{2n} \equiv (2 - 2^{2n})(-1) \equiv -1 \pmod{p}.$$

We have therefore

(4.3) $$pD_{2n} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid 2n) \\ 0 \pmod{p} & (p-1 \nmid 2n), \end{cases}$$

where p is any odd prime. We may accordingly state the following analog of the Staudt-Clausen theorem.

Theorem 3. *The number*

$$D_{2n} = 2^{2n}B_{2n}(\tfrac{1}{2})$$

satisfies

$$D_{2n} = G_{2n} - \sum_{\substack{p-1 \mid 2n \\ p > 2}} \frac{1}{p},$$

where G_{2n} *is an integer and the summation is restricted to odd primes such that* $p - 1 \mid 2n$.

This result is evidently a special case of Theorem 2.

5. Let

$$S_n = S_n(k) = \sum_{r=0}^{k-1} k^n.$$

Then, as we showed in paragraph 2,

$$S_n(k) = \sum_{r=0}^{n} \binom{k}{r+1} A_{nr},$$

so that

(5.1) $$\frac{1}{k} S_n(k) = \sum_{r=0}^{n} \frac{1}{r+1} \binom{k-1}{r} A_{nr}.$$

Then exactly as in paragraph 2, if $r + 1$ is composite and ≥ 6, the corresponding term in the right member of (5.1) is integral. Also if $r + 1 = 4$ and $n > 1$ the term is integral for n even, for n odd it follows from (2.5) that $(1/4)A_{n,s}$ is half of an odd integer. If $r + 1 = p$ we need only consider the

case $p-1 \mid n$. This yields

$$G - \frac{1}{p}\binom{k-1}{p-1} ,$$

where G is integral. Now the binomial coefficient $\binom{k-1}{p-1}$ is divisible by p except when $p \mid k$; in this case we have

$$\binom{k-1}{p-1} \equiv 1 \pmod{p} .$$

It therefore follows that (for $n > 1$)

$$(5.2) \qquad p\frac{S_n(k)}{k} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid n,\ p \mid k) \\ 0 \pmod{p} & (\text{otherwise}) , \end{cases}$$

As for the excluded case $n = 1$, we have

$$\frac{1}{k}S_1(k) = \frac{1}{2}(k-1) ,$$

so that (5.2) holds here also.

We may therefore state the following theorem which is also due to Staudt.

Theorem 4. *For n even*, $S_n(k)$ *satisfies*

$$(5.4) \qquad \frac{S_n(k)}{k} = G_n - \sum_{p-1 \mid n,\ p \mid k} \frac{1}{p}$$

where G_n *is integral and the sum extends over all primes p such that* $p-1$ *divides n and* $p \mid k$. *For n odd and* > 1, $S_n(k)/k$ *is integral.*

As a corollary of Theorem 4 we have

Theorem 5. *If k is divisible by every prime p such that* $p-1 \mid n$, *then*

$$\frac{S_n(k)}{k} - B_n$$

is integral for all $n \geq 1$.

Also if we combine Theorems 1, 2 and 4 we get

Theorem 6. *If h and k are relatively prime and* $k \geq 1$, *then*

$$b_n(h, k) + \frac{1}{k}S_n(k) = B_n + G_n ,$$

where G_n *is integral.*

6. We now introduce some notions concerning formal power series that are due to Hurwitz [7]. A series of the form

$$(6.1) \qquad \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} ,$$

where the a_n are arbitrary rational integers, is called a Hurwitz series, or briefly, an H-series. It follows at once that the sum, difference and product of two H-series is again an H-series. In particular if

$$(6.2) \qquad \sum_{n=0}^{\infty} \frac{b_n t^n}{n!}$$

is a second H-series, then the product of (6.1) and (6.2) is equal to

$$\sum_{n=0}^{\infty} \frac{c_n t^n}{n!} ,$$

where

$$c_n = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r} .$$

We notice also that

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{a_{n+1} t^n}{n!} , \quad \int_0^t \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \, dx = \sum_{n=1}^{\infty} \frac{a_{n-1} t^n}{n!} ,$$

so that the derivative and the definite integral of an H-series are also H-series.

For a series without constant term

$$\mathrm{H}_1(t) = \sum_{n=1}^{\infty} \frac{a_n t^n}{n!} ,$$

it follows from the identity

$$\frac{1}{k!} \mathrm{H}_1^k(t) = \int_0^t \mathrm{H}_1'(x) \frac{1}{(k-1)!} \mathrm{H}^{k-1}(x) \, dx$$

that

$$(6.3) \qquad \frac{1}{k!} \mathrm{H}_1^k(t)$$

is an H-series for all $k \geq 1$.

By the statement

$$\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \equiv \sum_{n=0}^{\infty} \frac{b_n t^n}{n!} \quad (\mathrm{mod}\ m)$$

is meant that the system of congruences

$$a_n \equiv b_n \quad (\mathrm{mod}\ m) \qquad (n = 0,\ 1,\ 2,\ \cdots)$$

is satisfied. This is equivalent to the assertion

$$\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!} + m\mathbf{H}(t) \,,$$

where $\mathbf{H}(t)$ is some H-series.

Thus the result concerning (6.3) can be stated in the form

(6.4) $$\mathbf{H}_1^k(t) \equiv 0 \quad (\mathrm{mod}\ k!) \,.$$

We shall apply these ideas to give another proof of the Staudt-Clausen theorem [11]. Consider the formula

$$\frac{t}{e^t - 1} = \sum_{r=0}^{\infty} \frac{1}{r+1} (e^t - 1)^r \,.$$

From (6.4) it follows that

$$\frac{(e^t - 1)^r}{r!}$$

is an H-series; therefore for $r+1 \geq 6$ and composite

$$\frac{1}{r+1} (e^t - 1)^r$$

is an H-series. Also by direct expansion

$$(e^t - 1)^3 = \sum_{n=3}^{\infty} (3^n - 3 \cdot 2^n + 3) \frac{t^n}{n!} \equiv 2 \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \quad (\mathrm{mod}\ 4) \,,$$

so that

(6.5) $$(e^t - 1) + \frac{1}{2}(e^t - 1)^3 \equiv t + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \quad (\mathrm{mod}\ 2) \,.$$

As for $r+1 = p$, we have

$$(e^t - 1)^{p-1} = \sum_{n=p-1}^{\infty} \frac{t^n}{n!} \sum_{s=0}^{p-1} (-1)^s \binom{p-1}{s} s^n$$

(6.6) $$\equiv \sum_{n=p-1}^{\infty} \frac{t^n}{n!} \sum_{s=0}^{p-1} s^n \equiv -\sum_{n=1}^{\infty} \frac{t^{n(p-1)}}{[n(p-1)]!} \quad (\mathrm{mod}\ p) \,.$$

The Staudt-Clausen theorem evidently follows from (6.5) and (6.6).
7. We shall now consider the class of H-series

$$(7.1) \qquad f(t) = \sum_{n=1}^{\infty} \frac{a_n t^n}{n!} \quad (a_1 = 1)$$

such that

$$(7.2) \qquad f'(t) = \sum_{n=0}^{\infty} A_n f^n(t) \quad (A_0 = 1) ,$$

where the A_n are integral. We shall show that for the numbers β_n defined by

$$(7.3) \qquad \frac{t}{f(t)} = \sum_{n=0}^{\infty} \frac{\beta_n t^n}{n!}$$

a theorem of the Staudt-Clausen type can be obtained.

It follows from (6.4) and (7.2) that

$$f'(t) \equiv \sum_{n=0}^{p-1} A_n f^n(t) \pmod{p} ,$$

where p is an arbitrary prime. Then it is clear that

$$(7.4) \qquad D^{p-1} f^{p-1}(t) \equiv \sum_{n=0}^{p-1} c_n f^n(t) \pmod{p} ,$$

where $D = d/dt$ and the c_n are integers. If we put

$$(7.5) \qquad f^{p-1}(t) = \sum_{n=p-1}^{\infty} \frac{a_n' t^n}{n!} ,$$

then since

$$f^{p-1}(t) f(t) \equiv 0 \pmod{p} ,$$

we get

$$(7.6) \qquad \sum_{r=0}^{n} \binom{n}{r} a_r a_{n-r}' \equiv 0 \pmod{p} \quad (n \geq p) .$$

For $n = p + 1$, (7.6) reduces to

$$(p+1) a_1 a_p' \equiv 0 \pmod{p} ,$$

so that

$$a_p' \equiv 0 \pmod{p} .$$

For $n = p + 2$ we get

$$(p+2)a_1 a_{p+1}' + \binom{p+2}{2} a_2 a_p' \equiv 0 \quad (\text{mod } p) ,$$

so that $a_{p+1}' \equiv 0 \pmod{p}$. Continuing in this way we get

$$a_p' \equiv a_{p+1}' \equiv \cdots \equiv a_{2p-3}' \equiv 0 \quad (\text{mod } p) .$$

Hence, using (7.4) and (7.5), it is seen that

(7.7) $$D^{p-1} f^{p-1}(t) \equiv -1 + c_{p-1} f^{p-1}(t) \quad (\text{mod } p) .$$

If in (7.6) we take $n = 2p - 1$ we get

$$(2p-1)a_1 a_{2p-2}' + \cdots + \binom{2p-1}{p} a_p a_{p-1}' \equiv 0 \quad (\text{mod } p) ,$$

which gives

$$a_{2p-2}' \equiv a_p a_{p-1}' \equiv -a_p \quad (\text{mod } p) .$$

Therefore (7.7) becomes

(7.8) $$D^{p-1} f^{p-1}(t) \equiv -1 + a_p f^{p-1}(t) \quad (\text{mod } p) .$$

As an immediate consequence of (7.8) we get

(7.9) $$f^{p-1}(t) \equiv -\sum_{n=1}^{\infty} a_p^{n-1} \frac{t^{n(p-1)}}{[n(p-1)]!} \quad (\text{mod } p) .$$

Now let

(7.10) $$\lambda(t) = \sum_{n=1}^{\infty} \frac{e_n t^n}{n!} \quad (e_1 = 1)$$

denote the inverse of the function (7.1), so that

(7.11) $$t = \sum_{n=1}^{\infty} \frac{e_n f^n(t)}{n!} .$$

Differentiating we get

$$1 = f'(t) \sum_{n=0}^{\infty} e_{n+1} \frac{f^n(t)}{n!} .$$

Comparison with (7.2) yields

(7.12) $$e_{n+1} \equiv 0 \quad (\text{mod } n!) ;$$

conversely (7.12) implies (7.2).

If we put

$$e_n' = \frac{e_n}{(n-1)!} \quad (n = 1, 2, 3, \ldots) ,$$

(7.11) becomes

$$t = \sum_{n=0}^{\infty} e_{n+1}' \frac{f^{n+1}(t)}{n+1} ,$$

so that

(7.13) $$\frac{t}{f(t)} = \sum_{n=0}^{\infty} e_{n+1}' \frac{f^n(t)}{n+1} .$$

If $n+1$ is composite and ≥ 6 it follows from (6.4) that

$$e_{n+1}' \frac{f^n(t)}{n+1}$$

is an H-series. If $n+1 = p$ we employ (7.8). We have therefore proved that if $p > 2$ and β_n is defined by (7.3), then

(7.14) $$p\beta_n \equiv \begin{cases} e_p a_p^{(n/(p-1))-1} \pmod{p} & (p-1 \mid n) \\ 0 \pmod{p} & (p-1 \nmid n) . \end{cases}$$

We show next that

(7.15) $$a_p + e_p \equiv 0 \pmod{p} .$$

Let $D_0^r g(t)$ denote the r-th derivative of $g(t)$ evaluated at $t = 0$. Then from (7.11)

(7.16) $$0 = \sum_{n=1}^{p} e_n D_0^p \frac{f^n(t)}{n!} .$$

Also

$$D_0^p(f^{k+1}(t)) = D_0^p(f^k(t)f(t)) \equiv f(0)D_0^p f^k(t) + f^k(0)D_0^p f(t) \pmod{p} ,$$

so that

$$D_0^p f^k(t) \equiv 0 \pmod{p} \quad (k > 1)$$

and (7.16) reduces to

$$e_p + D_0^p \frac{f^p(t)}{p!} \equiv 0 \pmod{p} .$$

But

$$D_0^p \frac{f^p(t)}{p} = D_0^{p-1}(f'(t)f^{p-1}(t)) \equiv f'(0)D_0^{p-1}f^{p-1}(t) + \cdots + f^{p-1}(0)D_0^p f(t) \equiv -1 \pmod{p}$$

and (7.15) follows at once.

Since

$$e'_p = \frac{e_p}{(p-1)!} \equiv -e_p \quad (\mathrm{mod}\ p),$$

(7.14) becomes

(7.17) $$p\beta_n \equiv \begin{cases} -e'^{n/(p-1)}_p & (\mathrm{mod}\ p) & (p-1\,|\,n) \\ 0 & (\mathrm{mod}\ p) & (p-1\nmid n). \end{cases}$$

The case $p = 2$ requires some further discussion. It follows from (7.8) that

$$f(t) \equiv \sum_{n=1}^{\infty} a_2^{n-1} \frac{t^n}{n!} \equiv t + a_2 \sum_{n=2}^{\infty} \frac{t^n}{n!},$$

$$\frac{1}{2} f^2(t) \equiv \sum_{n=2}^{\infty} a_2^{n-1} \frac{t^n}{n!} \equiv t + a_2 \sum_{n=3}^{\infty} \frac{t^n}{n!},$$

so that

$$f(t) \equiv t + a_2 \frac{f^2(t)}{2}.$$

This implies

$$\frac{1}{2} f^3(t) \equiv t \frac{f^2(t)}{2} \equiv \sum_{n=2}^{\infty} a_2^{n-2} \frac{t^n}{(n-1)!} \equiv \frac{t^3}{3!} + a_2 \sum_{n=2}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \quad (\mathrm{mod}\ 2).$$

Combining these congruences with (7.13), (7.14) and (7.17) we obtain the following theorem [3].

Theorem 7. *Let* $f(t)$ *be an* H*-series that satisfies* (7.2) *and define* β_n *by means of* (7.3). *Then for n even we have*

(7.18) $$\beta_n = G_n - \sum_{p-1\,|\,n} \frac{1}{p} e'^{n/(p-1)}_{\rightarrow},$$

while for n odd

(7.19) $$\beta_1 = \frac{1}{2} e'_2, \quad \beta_3 = G_3 + \frac{1}{2}(e'_2 + e'_4), \quad \beta_n = G_n + \frac{1}{2}(e'_2 + e'_2 e'_4),$$

where G_n *is integral and the summation in* (7.18) *is over all primes (including* 2) *such that* $p-1\,|\,n$.

8. If $f(t) = e^t - 1$, $\lambda(t) = \log(1+u)$, $e_n = (-1)^{n-1}(n-1)!$,

$$e'_n = (-1)^{n-1};$$

since $f'(t) = 1 + f(t)$, (7.2) is obviously satisfied. Clearly Theorem 7 is in agreement with the Staudt-Clausen theorem.

A much more interesting example is furnished by the special elliptic function $\phi(t)$ defined by

$$(8.1) \qquad \phi'^2(t) = 1 - \phi^4(t) , \quad \phi'(0) = 1 .$$

It follows from (8.1) that the inverse of $\phi(t)$ is

$$(8.2) \qquad \lambda(t) = \int_0^t \frac{du}{\sqrt{1 - u^4}} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{t^{4n+1}}{2^{2n}(4n+1)} .$$

The coefficients e_n' are determined by

$$(8.3) \qquad e_n' = \begin{cases} \dfrac{1}{2^{2r}} \dbinom{2r}{r} & (n = 4r+1) \\[2ex] 0 & \text{otherwise} ; \end{cases}$$

note that $e_2' = e_4' = 0$.

We put

$$(8.4) \qquad \frac{t}{\phi(t)} = \sum_{n=0}^{\infty} \frac{F_n t^{4n}}{(4n)!} .$$

It is not difficult to show that

$$(8.5) \qquad F_n = 2^{2n} F_n' , \quad F_n' \equiv (-1)^n \pmod{4} .$$

For brevity we shall however omit the proof of (8.5).

Applying Theorem 7, we obtain

$$(8.6) \qquad F_n = G_n - \sum_{p-1 \mid 4n} \frac{1}{p} e_p'^{\,4n/(p-1)} ,$$

where G_n is integral and the summation is over primes of the form $4k+1$ only. By a theorem of Gauss [1, p. 137] we have

$$\epsilon_p = \frac{1}{2^{2k}} \binom{2k}{k} \equiv \frac{3 \cdot 7 \cdot 11 \cdots (4k-1)}{1 \cdot 5 \cdot 9 \cdots (4k-3)} \equiv 2a \pmod{p} ,$$

where $p = 4k+1$ and the odd integer a is uniquely determined by

$$p = a^2 + b^2 , \quad a \equiv b+1 \pmod{4} .$$

Thus (8.6) becomes

$$(8.7) \qquad F_n = G_n - \sum_{p-1 \mid 4n} \frac{1}{p} (2a)^{4n/(p-1)} .$$

This result is due to Hurwitz [7], who proved it by making use of the

complex multiplication of the function $\phi(t)$.

9. In conclusion we add a few words about certain sequences of rational numbers for which a theorem of the Staudt-Clausen type either does not hold or has not been obtained. Of particular interest are the Bernoulli numbers of order k [9, Chapter 6] defined by

$$\left(\frac{t}{e^t-1}\right)^k = \sum_{k=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.$$

For certain values of n and k the denominator of $B_n^{(k)}$ may be divisible by arbitrarily high powers of a prime. For example

$$p^r B_n^{(k)} \equiv (-1)^r \pmod{p},$$

where $k = (p^r-1)/(p-1)$ and $n = p^{r-1}(s(p-1)+1)-1$. Also it has been proved [4] that if

$$k = a_1 p^{i_1} + a_2 p^{i_2} + \cdots + a_r p^{i_r} \quad (0 \leqq i_1 < i_2 < \cdots < i_r; \; 0 \leqq a_s < p)$$

then $p^r B_n^{(k)}$.

An example (suggested by Theorem 3) about which nothing is known is

$$t \left\{ \sum_{n=0}^{\infty} \frac{t^{3n+1}}{(3n+1)!} \right\}^{-1} = \sum_{n=0}^{\infty} \frac{3n\, t^{3n}}{(3n)!};$$

the 3 may of course be replaced by any integer $k > 2$.

Again nothing is known about the coefficients defined by

$$\frac{t^2}{e^t-1-t} = \sum_{n=0}^{\infty} \frac{\beta_n t^n}{n!}.$$

One can easily construct additional examples.

A problem of somewhat different appearance is suggested by the Bessel function of order 1.

$$t \left\{ \sum_{n=0}^{\infty} \frac{t^{2n+1}}{n!\,(n+1)!} \right\}^{-1} = \sum_{n=0}^{\infty} \frac{\beta_n t^{2n}}{n!\,n!}.$$

Comparing these examples with Theorem 7, the difficulty seems to be that no explicit formula is available for the coefficients of the inverse function.

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Duke University
Durham, North Carolina

SIMILARITY CLASSIFICATIONS OF COMPLEX MATRICES

John S. Lomont, John E. Maxfield and R. G. Selfridge

INTRODUCTION: This paper concerns itself with matrices under similarity transformations. Unless stated otherwise all matrices are assumed to have entries from the complex field.

Definition 1. A matrix A is said to be similar to a matrix B if there exists a non-singular matrix P such that $PAP^{-1} = B$.

Definition 2. A matrix A is said to be diagonalizable if it is similar to a matrix having only zero elements off the principal diagonal.

Theorems 1, 2 and 3 are known and appear in the literature. For ease of reading and completeness in this note, simple proofs are included. These theorems are needed for the proofs of Theorems 4 and 5 which do not appear in the literature.

Theorem 1. Two diagonalizable matrices A and B are similar if and only if:

(1) the dimension of A, written $\dim(A) = \dim(B) = n$

(2) the trace of A^j, written $T(A^j) = T(B^j)$, $j = 1, 2, \cdots, n$.

Proof: Assume A is similar to B so that $B = SAS^{-1}$. Then, $\dim(A) = \dim(B)$ and

$$T(B) = T(SAS^{-1}) = T(A).$$

Further,

$$T(B^j) = T((SAS^{-1})^j) = T(SA^jS^{-1}) = T(A^j).$$

Assume $\dim(A) = \dim(B)$ and $T(A^j) = T(B^j)$, $j = 1, 2, \cdots, n$. Let $C = SAS^{-1}$ and $D = TBT^{-1}$ be the Jordan canonical forms of A and B. Then A is similar to B if and only if C is similar to D. But

$$C = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad D = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix},$$

and

$$C^j = \begin{bmatrix} \lambda_1^j & & & \\ & \lambda_2^j & & \\ & & \ddots & \\ & & & \lambda_n^j \end{bmatrix}, \quad D^j = \begin{bmatrix} \mu_1^j & & & \\ & \mu_2^j & & \\ & & \ddots & \\ & & & \mu_n^j \end{bmatrix}.$$

$$T(C^j) = \sum_{i=1}^{n} \lambda_i^{\,j} \quad \text{and} \quad T(D^j) = \sum_{i=1}^{n} \mu_i^{\,j} \,.$$

But $T(C^j) = T(A^j) = T(B^j) = T(D^j)$. Hence

$$\sum_{i=1}^{n} \lambda_i^{\,j} = \sum_{i=1}^{n} \mu_i^{\,j}, \quad j = 1, 2, \cdots, n \,.$$

But by elementary symmetric function theory this implies that $\lambda_i = \mu_j$ in some order (Newton's formulas [2, P. 261]). Since the order of the diagonal elements of D can be changed by similarity transformation, C is similar to D and thus A is similar to B.

Corollary. If the traces of the corresponding matrices in two matric representations of the same dimension of a finite cyclic group are equal, then the two representations are similar.

Note 1. If A and B are not diagonalizable the conditions of the theorem are not sufficient, even if non-singularity is added.

Let

$$A = \begin{bmatrix} 0 & -\dfrac{a^2}{4} \\ 1 & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \dfrac{a}{2} & 0 \\ 0 & \dfrac{a}{2} \end{bmatrix}, \quad a \neq 0 \,.$$

Then

$$A^2 = \begin{bmatrix} \dfrac{-a^2}{4} & \dfrac{-a^3}{4} \\ a & \dfrac{3\,a^2}{4} \end{bmatrix} \quad \text{and} \quad B^2 = \begin{bmatrix} \dfrac{a^2}{4} & 0 \\ 0 & \dfrac{a^2}{4} \end{bmatrix} \,.$$

Thus $T(A^j) = T(B^j)$, $j = 1, 2$. Assume there exists a non-singular matrix P such that $PA = BP$. Then

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} 0 & \dfrac{-a^2}{4} \\ 1 & a \end{bmatrix} = \begin{bmatrix} \dfrac{a}{2} & 0 \\ 0 & \dfrac{a}{2} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \,.$$

This implies that

$$\frac{-p_1 a}{2} + p_2 = 0 \quad \text{and} \quad \frac{-p_3 a}{2} + p_4 = 0 \,.$$

Thus P is singular, establishing the note.

Theorem 2. A diagonalizable complex matrix C is similar to a real matrix R if and only if its complex characteristic roots appear in conjugate pairs. ·

Proof: Certainly, if C is similar to a real matrix its characteristic equation has real coefficients and the complex characteristic roots appear in conjugate pairs.

If the complex characteristic roots of C occur in conjugate pairs and C is diagonalizable, there exists a non-singular matrix S such that $SCS^{-1} = D$, a diagonal matrix, where conjugate pairs of complex characteristic roots are located together down the diagonal.

$$D = \begin{bmatrix} \lambda_1 & & & & & & \\ & \overline{\lambda}_1 & & & & 0 & \\ & & \lambda_2 & & & & \\ & & & \overline{\lambda}_2 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & \lambda_k \\ & & & & & & & \overline{\lambda}_k \\ & & & & & & & & R' \end{bmatrix},$$

where R' is a real diagonal matrix. Let

$$T = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k \dotplus I,$$

where

$$T_j = \begin{bmatrix} \overline{\lambda}_j & \lambda_j \\ 1 & 1 \end{bmatrix}.$$

Then TDT^{-1} is

$$R = D_1 \dotplus D_2 \dotplus \cdots \dotplus D_k \dotplus R',$$

where

$$D_j = \begin{bmatrix} 0 & \lambda_j \overline{\lambda}_j \\ -1 & \lambda_j + \overline{\lambda}_j \end{bmatrix},$$

which is real. Thus $TSCS^{-1}T^{-1} = R$, proving the theorem.

Theorem 3. A necessary and sufficient condition that an n by n diagonalizable matrix C be similar to a real matrix R is that the traces of the first n powers of C be real.

Proof: By Theorem 2, if C is similar to a real matrix its roots appear in conjugate pairs and thus their sum, the trace of C, is real. If C is similar to a real matrix, so is C^j. Thus necessity is proved.

If C is diagonalizable and if the traces of the first n powers of C are real, then

$$S_j = \sum_{i=1}^{n} \lambda_i^j$$

can be expressed as a polynomial in the elementary symmetric functions of

the characteristic roots of C in the following way:

$$S_j = -c_1 S_{j-1} - c_2 S_{j-2} - \cdots - j c_j$$

or $S_1 + c_1 = 0$, $S_2 + c_1 S_1 + 2 c_2 = 0$, etc. (Newton's formulas), where the C_j are the coefficients of the characteristic equation

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0 .$$

By assumption S_1 is real, thus so is c_1; S_2 is real, thus so is c_2; $\cdots$. Therefore, the coefficients of the characteristic equation of C are real and by theorem 2 C is similar to a real matrix R.

Definition 3. An algebraic integer is a root of an equation of the form

$$x^s + a_1 x^{s-1} + \cdots + a_s = 0$$

where the a_i are rational integers.

Theorem 4. A necessary and sufficient condition that an n by n diagonalizable matrix A be similar to a matrix having only rational integral entries is that the characteristic roots of A be algebraic integers and that the traces of the first n powers of A be rational integers.

Proof: If A is similar to a matrix having only rational integral entries, since the trace and characteristic function are similarity invariants, the traces of all powers are rational integers and the characteristic equation is of integral type. Thus, the characteristic roots are algebraic integers.

If A is diagonalizable and its characteristic roots, the λ_i, are algebraic integers, and if $T(A^j) = r_j$ are rational integers for $j = 1, 2, \cdots, n$, then

$$T(A^j) = \sum \lambda_i^j = r_j ,$$

and the coefficients of the characteristic equation $f(x) = 0$ are algebraic integers and, by symmetric function theory, rational numbers — thus, rational integers. Factor $f(x)$ into its irreducible polynomial factors over the rationals,

$$f(x) = f_1(x) f_2(x) \cdots f_k(x) = 0 .$$

Each of these polynomials has rational integral coefficients. Since A can be transformed into its diagonal form

$$D = \begin{bmatrix} \lambda_{11} \\ & \lambda_{12} \\ & & \cdot \\ & & & \cdot \\ & & & & \lambda_{1s_1} \end{bmatrix} \dotplus \begin{bmatrix} \lambda_{21} \\ & \lambda_{22} \\ & & \cdot \\ & & & \cdot \\ & & & & \lambda_{2s_2} \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} \lambda_{k1} \\ & \lambda_{k2} \\ & & \cdot \\ & & & \cdot \\ & & & & \lambda_{ks_k} \end{bmatrix}$$

$$= D_1 \dotplus D_2 \dotplus \cdots \dotplus D_k ,$$

where the λ_{ij} are the roots of

$$f_i(x) = x^{s_i} + a_{i1}x^{s_i-1} + \cdots + a_{is_i} = 0 \ ,$$

we can consider only the blocks D_i, and work on these one at a time. By [1, p. 73], D_i is similar to

$$\begin{bmatrix} -a_{i1} & -a_{i2} & \cdots & -a_{is-1} & -a_{is} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

all of whose entries are rational integers.

Theorem 5. A necessary and sufficient condition that a matrix be similar to a real matrix is that it be similar to its complex conjugate.

Proof: If a matrix is similar to a real matrix it is certainly similar to its complex conjugate.

Let us assume that A is in its Jordan canonical form and that A is similar to its complex conjugate. This implies that the Segré characteristics associated with the characteristic root λ_i are the same as those associated with the root $\bar{\lambda}_i$. Thus, we can rearrange the direct sum that forms the Jordan canonical form into adjacent pairs of matrices that differ only in that λ_i is replaced by $\bar{\lambda}_i$ – or A_i by $\bar{A}_i$. Let us consider A as a direct sum

$$A = (A_1 \dotplus \bar{A}_1) \dotplus (A_2 \dotplus \bar{A}_2) \dotplus \cdots$$

and investigate $A_i \dotplus \bar{A}_i$.

$$A_i = \begin{bmatrix} \lambda_i & & & & 0 \\ 1 & \lambda_i & & & \\ & 1 & \lambda_i & & \\ & & 1 & \cdot & \\ 0 & & & & \cdot \end{bmatrix} \qquad \bar{A}_i = \begin{bmatrix} \bar{\lambda}_i & & & & 0 \\ 1 & \bar{\lambda}_i & & & \\ & 1 & \bar{\lambda}_i & & \\ & & 1 & \cdot & \\ 0 & & & & \cdot \end{bmatrix}$$

where the λ_i are not real.

Note that $A_i\bar{A}_i = \bar{A}_iA_i$ is real and $A_i - \bar{A}_i$ is non-singular.

Let

$$P_i = \begin{bmatrix} \bar{A}_i & A_i \\ I & I \end{bmatrix} \cdot P_i^{-1} = \begin{bmatrix} (\bar{A}_i - A_i)^{-1} & -A_i(\bar{A}_i - A_i)^{-1} \\ -(\bar{A}_i - A_i)^{-1} & \bar{A}_i(\bar{A}_i - A_i)^{-1} \end{bmatrix}$$

is a right inverse of P and thus the inverse of P and

$$P_i(A_i \dotplus \overline{A}_i)P_i^{-1} = \begin{bmatrix} 0 & A_i\overline{A}_i \\ -I & A_i + \overline{A}_i \end{bmatrix},$$

which is real.

Note 2. The matrix must be diagonalizable for the conditions of Theorems 2, 3 and 4 to be sufficient.

Let

$$A = \begin{bmatrix} i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

n A has characteristic equation $(x^2+1)^2 = 0$, and

$$A^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2i & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -i & 0 & 0 & 0 \\ -3 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}, \quad \text{and} \quad A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 5 for A to be similar to a real matrix it must be similar to

$$\overline{A} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

However, this is impossible, since A and $\overline{A}$ do not have the same elementary divisors and thus Segré characteristics.

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International Business Machines Corporation
Naval Ordnance Test Station
Miami University

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

WE SHOULD TEACH OUR STUDENTS *ABOUT* MATHEMATICS

Louis R. McCreery

With all the new mathematics which has become important in recent years, and all the excellent new curricula which are being developed, we in mathematics education are not too unlike a small boy I observed recently. He was facing the dessert counter of a fancy cafeteria with only fifteen cents in his pocket. In his case, as in ours, the process of choosing is most difficult.

To make good choices, guidelines are needed. These guidelines are emerging in such statements as these, "We should teach for the understanding of the nature and role of mathematics", or, "It is important that mathematics be so taught that students will, in later life, be able to learn new mathematical skills which the future will surely demand of them."

Such statements indicate that we are working toward a basic philosophy, which is more necessary than usual in times of rapid change. For thorough understanding and maximum use, our students must have a sound idea of "the nature and role" of mathematics.

Here is a sample philosophy, developed to provide such guide lines for a mathematics department, particularly in the study of geometry.

A Philosophy of Mathematics

Mathematics may be described as a method of investigation and a body of knowledge made up of interrelated systems. Each of these systems contains the following parts, and wherever possible the ideas behind these parts are expressed thru symbols.

1. Carefully delineated elements with the exact meanings expressed through undefined terms, defined terms, and postulates.

2. Operations and postulates which describe how these elements can be combined, compared, and otherwise manipulated.

3. Deductions which logically follow through the indicated manipulation of these elements. These deductions make up the body of mathematical truth. Much of modern mathematics seems to differ from Bertrand Russell's viewpoint "mathematics may be defined as the subject in which we

never know what we are talking about, nor whether what we are saying is true."

In algebra, the elements are numbers, the operations are addition, subtraction, multiplication, division, comparison, and substitution. The deductions are such things as formulae, functions, and solutions. In set theory, the elements are sets, the operations intersection, union, and complementation. Other branches of mathematics exhibit a similar sort of structure.

Geometry fits this pattern and provides perhaps the best opportunity to present the interrelations and the variety of mathematical systems. Here the elements are point, line, and plane. Most of the postulates, as with Euclid's fifth, serve to define the elements and the space to which they are limited. Some truth is discovered and established through operations on these elements by construction, combination, and superposition. More comes to light when the algebraic operations are used on these non-number elements.

Some of the excitement of geometry comes when we make changes in the elements or in the definition of space. If parallel lines do meet, we have a different type of plane surface. We have a new but related geometry, with different conclusions and different applications. If we give direction to a Euclidean line segment, a vector is formed, and in this geometry, the sum of two sides of a triangle is equal to the third side. If one is to face an unknown mathematical situation, he must know that changes like these can be made, under what circumstances they are valid, and what new conclusions can be drawn.

With the introduction of number, measure, and coordinates into the study of geometric elements, the interrelation of geometry with other systems of mathematics becomes apparent. Algebra is most helpful in establishing geometric truth, and geometry is helpful to algebra in return. Part of the awe and wonder of our subject stems from the fact that different systems give the same answer to basic problems. The great strength of mathematics lies in its consistency, which to this author is indicative of the underlying order of the universe.

Mathematical Truth

Mathematical truth is *discovered* inductively. Such discovery requires imagination and insight. This truth is *established* deductively, requiring rigor, care, and once more imagination and insight. Geometry is a prime example of this inductive-deductive process. The Egyptians invented the "rope-swinging" constructions and used them because they seemed to give the right answers. The Greeks developed a deductive system to prove that the answers were right. In geometry the student can be made aware of this process as a general approach, and should be given the opportunity to develop the traits and understandings necessary to adapt the process to his own use. Mathematicians must be open-minded about possibilities, tough-minded about proof.

Applications of Mathematics

For a long time, we have considered the ability to translate "word problems" or a given physical situation into algebra and calculus as one of the top tests of skill. The difficulty has been compounded by the many new systems and applications. Now we must know how to choose which one of many systems most closely parallels the situation we are investigating, and perhaps develop some variations of our own. A simple example: to lay out the airport for the Air Force Academy, Euclidean plane geometry was required; to lay out the boundaries of the State of Colorado, the use of spherical geometry, a non-Euclidean form, was required. Mathematically, they are different but related systems. Their chief difference lies in the delineation of the plane. Traditional geometry provides a mathematical model for the flat airport. Spherical geometry provides a model for the surface of a large area of the earth.

Mathematics and Other Fields of Knowledge

Philosophers are envious of mathematicians, for, as Kant implies, mathematics starts with carefully defined elements and makes its conclusions from these. It abstracts from life the elements with which it can work, while philosophy must take life as a whole. The latter is fortunate to end up with a good definition, which is the starting point for mathematics.

Science deals with facts observed. Most of its facts are inductive, but after the elements and laws of a given science are determined, a closely parallel mathematical system is chosen or developed to aid in further investigation.

Art and mathematics, particularly geometry, make for interesting comparisons. The approach of the artist is highly intuitive and subjective, while the mathematician prides himself on his utter objectivity. It is interesting to note that they often come up with the same results, as in the case of the "golden section," indicating a basic consistency in all knowledge. The purely intuitive choice of the dimensions of the most beautiful rectangle is found to agree with a rigorous mathematical idea.

The foregoing comments are examples of the kind of ideas we need about mathematics. We should develop methods to bring them alive in the minds of our students. Each of us needs his own working definition of the subject and some idea of its relation to other fields of knowledge. We need such ideas to guide us in the present growth and resulting confusion, and our students need them as they face new systems and new situations.

Without a basic philosophy, we become too much like blind men leading the blind.

5361 Candace Pl.
Los Angeles 41, Calif.

KEEP THE SIGNLESS NUMBERS

William R. Ransom

If the algebra class is not taught to distinguish signless from positive numbers, at least two important principles are damaged:

1. Unique factorization: but $-6 = -2 \times +3$ or $+2 \times -3$.

2. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$: but $\sqrt{-2} \times \sqrt{-3} = -\sqrt{+6}$.

Annoying qualifications have to be annexed to these principles if they are not restricted to signless numbers.

Originally $+$ and $-$ were warehouse marks, with a meaning apart from the numbers to which they were attached: $+2$ meant 2 lbs. over-weight, and -2 meant 2 lbs. under-weight.

When the algebra people took over and welded the sign to the number, they did not see that they were sacrificing an important distinction: the difference between magnitude and direction. Magnitude was lost: in its place appeared *order*. People put on signs, and then to get them off introduced "absolute magnitude".

Temperature is a scalar. Two temperatures have order and not magnitude. Because from $+2$ to $+3$ on a scale procedes in the positive direction from a smaller to a larger number, many confound $4 < 3$ with $+2 < +3$, and read the sign "is less than" in both cases. This confusion of magnitude with order leads them to read $-1 < 0$, as "minus one is less than zero", a paradox that ought to be kept out of mathematics: what it means is "-1 precedes 0". The words *precede* and *follow* should be used when Order is in question. Surely it is not too late to gegin saying these things right. Should algebra students be asked to accustom themselves to locutions which do not mean what they say?

When it comes to complex numbers, j (preferred to i because i suggests the derogatory word "imaginary") is used in exactly the way that it is here proposed to use $+1$ and -1. It would be consistent to keep the signless numbers, and extend the number system of algebra by the introduction of the multipliers $+1$, -1, and j, rather than the whole array of positive and negative numbers.

Tufts University

A CLASSROOM PRESENTATION OF THE DEFINITE INTEGRAL

Jerome Manheim

The elementary calculus student must make peace with a trilogy of ideas about the definite integral. On the one hand it represents the area under a curve; again, it is the limit of a sum; and finally, it is a particular anti-derivative. After the definite integral is defined in terms of one of these concepts its relatedness to the other concepts needs to be argued.

Some authors define area as the limit of the sum but then never explain what this area has to do with conceptualized area. Nor does this induce a reticence to use the integral to find such conceptualized areas as, for example, in finding the work performed as the area under a curve in the distance-force space. Alternate approaches, appealing to the geometrically obvious, assert that the error, which derives from approximating an area by rectangles, vanishes in the limiting process. If this is indeed intuitively evident the obviousness must at best result from consideration of a particular partitioning of a particular curve. The object of this note is to propose a sequential development which does not impose upon the student the need "to see" that the error is approaching zero. In this way the obviousness of the argument can be made independent of special curves or special partitionings. Existence theorems for the definite integral will be assumed, in particular, the theorem that the integral of a continuous function exists.

Let a single-valued, continuous function, $y = f(x)$, be defined over the interval $a \leq x \leq b$. Partition the interval into n parts, denoting the partition points by $x_0 = a$, x_1, x_2, $\cdots$, $x_n = b$, where $x_i < x_{i+1}$. Let $\overline{x}_k$ be a value of x such that $x_{k-1} \leq \overline{x}_k \leq x_k$. Form the number

$$G = \lim_{\max(x_k - x_{k-1}) \to 0} \sum_{k=1}^{n} f(\overline{x}_k)(x_k - x_{k-1}) .$$

If G exists and is independent of the partitioning and of the manner in which $\overline{x}_k$ is chosen then G is defined as the definite integral of $f(x)$ from a to b. So that this can be made more meaningful an example of a function not integrable in this (Riemann) sense can be given. The function which is zero for the irrationals and one for the rationals is an elementary example, yielding one in the limiting process if $\overline{x}_k$ is chosen as to be always rational, zero if $\overline{x}_k$ is always irrational. (The integral is defined in the Lebesgue sense and has the value zero.)

The next step is to prove the Law of the Mean for Integrals. It should be noted that the proof of this theorem is strictly analytic and that any graphical interpretations are not part of the development. This is crucial

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because the Law asserts only that the value of an integral is a number numerically equal to the measure of the area of a rectangle and not that the integral represents an area under a curve. The later introduction of Improper Integrals in the usual limit-taking manner will allow an extension of the definition of the integral to a piecewise-continuous function, without offending either the Law of the Mean or the area idea we are trying to preserve, since the point(s) of discontinuity are only approached.

The Law of the Mean for Integrals can now be employed to demonstrate the relationship between integration and differentiation. This is a standard development.

To this point developments have been entirely analytic. The last part of the program is to correlate these results with the geometric notion of area.

Consider the configuration bounded by the graph of the function $f(x)$, positive and continuous for $a \leq x \leq b$, the x-axis, and the lines $x = a$ and $x = b$. Each such configuration has a "mind's eye" interpretation of content called the area, and it is this conceptual area that is to be identified with the definite integral. It is necessary to impose a single, reasonable, constraint upon the individual, and possibly diverse, interpretations, namely, that the area be bounded above (below) by the area of the circumscribed (inscribed) rectangle. Let the interval from a to b be divided into n parts and erect ordinate lines at the points of division. Then, each sub-area is similarly bounded above and below by the circumscribed and inscribed rectangles. In each interval $(x_k - x_{k-1})$ the length of the ordinate is a continuous function of the abscissa and hence the areas defined by the product of the ordinates and the length of the interval are a continuous function. Since the areas assume all values between the areas of the inscribed and circumscribed rectangles, inclusive, there exists a point x_k' in the interval such that the area of the rectangle determined by it is the same as the area under the curve segment. When this construction is made in each of the n intervals, the area of the n rectangles exactly equals the area under the curve. It remains only to pass to the limit always retaining the equality of areas, and this limit is, by definition, the definite integral. No generality is lost by the given partitioning of the given curve that is used for illustrative purposes because the equality of areas is maintained. The Law of the Mean will, of course, generate the same rectangles but its application here is proscribed for the reason stated before.

Montclair State College
Upper Montclair, New Jersey

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

TERMINAL DIGITS OF $MN(M^2 - N^2)$

C. W. Trigg

If M and N are integers, the unit's digit of

$$P = MN(M^2 - N^2) = MN(M+N)(M-N)$$

is dependent upon the unit's digits of its four factors. Let the unit's digits of M, N, P be m, n, p, respectively. Now p will be zero if: $m = n$; or m, n, or $(m-n)$ equals 0 or 5; or $m + n$ ends in 0 or 5.

These zeros form a symmetrical pattern in the square array of the values of p. Thus

n \ m	1	2	3	4	5	6	7	8	9
1	0	6	4	0	0	0	6	4	0
2	6	0	0	6	0	4	0	0	6
3	4	0	0	4	0	6	0	0	4
4	0	6	4	0	0	0	4	6	0
5	0	0	0	0	0	0	0	0	0
6	0	4	6	0	0	0	6	4	0
7	6	0	0	4	0	6	0	0	6
8	4	0	0	6	0	4	0	0	4
9	0	6	4	0	0	0	6	4	0

The non-zero elements may be filled in by direct computation. Or, the following properties of the array may be used:

1) The array is symmetrical to the principal diagonal (where $m = n$), since interchange of M and N merely changes the sign of P.

2) In the upper-right triangle of the array (where $m > n$), $p_{m,n} = p_{m+5,n} = p_{m,n+5}$ because

$$p_{m+5,n} = (m+5)n[(m+5)^2 - n^2] = mn(m^2 - n^2) + 5n[3m(m+5) - n^2 + 25] .$$

The second term is positive and is a multiple of 10 since $m(m+5)$ is even.

3) Elements in the upper-right triangle and symmetrical to the perpendicular bisectors of the sides (where $m = 5$ or $n = 5$) are complementary.

That is, $p_{5-q,n} + p_{5+q,n} = 10$ and $p_{m,5-r} + p_{m,5+r} = 10$, since

$$(5-q)n[(5-q)^2 - n^2] + (5+q)n[(5+q)^2 - n^2] = 10n(25 + 3q^2 - n^2).$$

4) Elements symmetrical to the anti-diagonal, which runs from upper right to lower left, are complementary. This follows since

$$p + p_c = mn(m^2 - n^2) + (10-n)(10-m)[(10-n)^2 - (10-m)^2]$$

$$= 10(m-n)[(10-m-n)(20-m-n) + 2mn],$$

where p_c is the complement of p.

By using these four properties, all of the non-zero elements of the array may be filled in, once $p_{2,1} = 6$ is established.

Some by-product observations:

a) The elements symmetrical to the central element, $p_{5,5}$, are complementary.

b) On either side of the principal diagonal, a continuous path of Knight's moves joins the 6's. Another Knight's-move path joins the 4's. The 4's path on one side of the diagonal joins with the 6's path on the other side to form a closed path. The joins of the elements in one closed path form an expanded right-hand swastika, and the joins of the elements in the other path form an expanded left-hand swastika.
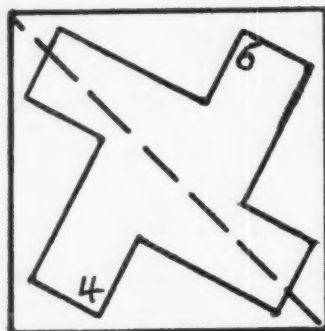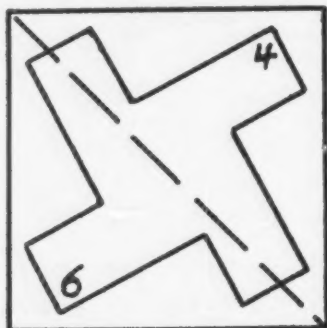
c) The sum of the elements in rows (or columns) k and $10-k$ is 40, $k \leq 4$.

d) If two integers, M and N, are chosen at random, the probabilities that P will end in 0, 4, or 6 are 0.68, 0.16, or 0.16, respectively.

Los Angeles City College

THE EVALUATION OF SUMMATIONS WITH
BINOMIAL COEFFICIENTS

F. S. Nowlan

1. *Introduction.* This paper deals with the evaluation of summations of the form

$$(1) \quad \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} (a_1 + it_1)^{p_1} (a_2 + it_2)^{p_2} (a_3 + it_3)^{p_3} \cdots (a_j + it_j)^{p_j}, \quad t \neq 0,$$

in which, subject to the restriction $p_1 + p_2 + \cdots + p_j = w \leq n$, the p's may denote any *positive* integers. On the other hand, the a's and t's may take on any values.

We refer to n as the *order* of the summation and w as its *weight*.

It will be shown that in case $w < n$, the summation (1) has the value *zero*. Also, if $w = n$, the value of the summation is $(-1)^n (n!) T$, where $T = t_1^{p_1} t_2^{p_2} \cdots t_j^{p_j}$, which is independent of the a's.

We observe that in an extreme case, the summation (1) takes the form

$$(2) \quad \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} (a + it)^w, \quad w \leq n .$$

It will be noted that the terms of the summation (1) are the products of the *corresponding* terms of the sequences $\sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1}$ and the j sequences $\sum_{i=1}^{n+1} (a_k + it_k)^{p_k}$, $(k = 1, 2, \cdots, j)$. The terms of $\sum_{i=1}^{n+1} (a_k + it_k)^{p_k}$, are either in arithmetic progression, the case if $p_k = 1$, or are powers of expressions $a_k + it_k$. We refer to these latter as *basic factors* of the terms of the sequence $\sum_{i=1}^{n+1} (a_k + it_k)^{p_k}$. The *basic factors* are always in arithmetic progression.

2. *The Reduction of the Summation* (1) *to a Simpler Form.* The term of the summation (1) obtained by placing $i = 1$ will be referred to as *the initial term.* It serves as a guide for the expansions of the other terms. Thus, the expansion of the i-th term of the summation may be obtained from that of *the initial term* by the replacement of the *increments* t_1, t_2, $\cdots$, t_j by it_1, it_2, $\cdots$, it_j, followed by the multiplication of the resulting sum by the

binomial coefficient $(-1)^{i-1} \binom{n}{i-1}$.

We shall be concerned with *addition by columns*. To illustrate our meaning, we consider the case $n = 4$ and the summation

$$\sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1}(a+it)^4 \, , \quad t \neq 0 \, .$$

The terms of the summation are

$$1(a+t)^4 = 1[a^4 + 4a^3t + 6a^2t^2 + 4at^3 + t^4] \, ,$$

$$-4(a+2t)^4 = -4[a^4 + 4a^3(2t) + 6a^2(2t)^2 + 4a(2t)^3 + (2t)^4] \, ,$$

$$+6(a+3t)^4 = +6[a^4 + 4a^3(3t) + 6a^2(3t)^2 + 4a(3t)^3 + (3t)^4] \, ,$$

$$-4(a+4t)^4 = -4[a^4 + 4a^3(4t) + 6a^2(4t)^2 + 4a(4t)^3 + (4t)^4] \, ,$$

$$+1(a+5t)^4 = +1[a^4 + 4a^3(5t) + 6a^2(5t)^2 + 4a(5t)^3 + (5t)^4] \, .$$

Then, upon an addition by columns, we obtain:

Col. 1. $1a^4(1 - 4 + 6 - 4 + 1) = 0 \, ,$

Col. 2. $4a^3t(1 \cdot 1 - 4 \cdot 2 + 6 \cdot 3 - 4 \cdot 4 + 1 \cdot 5) = 0 \, ,$

Col. 3. $6a^2t^2(1 \cdot 1^2 - 4 \cdot 2^2 + 6 \cdot 3^2 - 4 \cdot 4^2 + 1 \cdot 5^2 = 0 \, ,$

etc..

It will be observed that in *the addition by columns* of the expansions of the terms of the summation (1), there is one column, the last, whose sum is independent of the a's. This sum is

$$1 \cdot [t_1^{p_1} t_2^{p_2} \dots t_j^{p_j}] - \binom{n}{1}[(2t_1)^{p_1}(2t_2)^{p_2} \dots (2t_j)^{p_j}] + \binom{n}{2}[(3t_1)^{p_1}(3t_2)^{p_2} \dots (3t_j)^{p_j}] \dots$$

$$+ (-1)^n \binom{n}{n}[\{(n+1)t_1\}^{p_1}\{(n+1)t_2\}^{p_2} \dots \{(n+1)t_j\}^{p_j}] \, .$$

We denote the product $t_1^{p_1} t_2^{p_2} \dots t_j^{p_j}$ by T. Then, if we factor out T from the various terms, and recall that $p_1 + p_2 + \dots + p_j = w \leqq n$, we can write the sum of the terms in the last column in the form

(3) $T[1^w - \binom{n}{1}2^w + \binom{n}{2}3^w - \dots + (-1)^n \binom{n}{n}(n+1)^w] \, , \quad w \leqq n \, .$

On the other hand, the remaining columns consist of terms which contain one, or more, of the a's and accordingly the number of their t's is fewer than n. We illustrate the sum for such a column, choosing $j \geqq 5$:

(4) $(a_2^{p_2} a_4^{p_4} t_1^{p_1} t_3^{p_3} t_5^{p_5} \dots t^{p_j})[1^w - \binom{n}{1}2^w + \binom{n}{2}3^w - \dots + (-1)^n \binom{n}{n}(n+1)^w] \, ,$

with $w = p_1 + p_3 + p_5 + \dots + p_j < n \, .$

It becomes evident that the expression

(5) $$1^w - \binom{n}{1}2^w + \binom{n}{2}3^w - \cdots + (-1)^n \binom{n}{n}(n+1)^w$$

plays an important part in determining the value of the summation (1).

It is well known that the summation (5) has the value $(-1)^n(n!)$ for $w = n$, and equals *zero* for $w < n$. (A proof is outlined in Ex. 2, p. 259, of Hall and Knight's *Higher Algebra*, MacMillan and Co., Ltd.) It follows that the summation (1) has the value *zero* for the weight $w < n$, and equals $T(-1)^n \cdot (n!)$, $T = t_1^{p_1} t_2^{p_2} \cdots t_j^{p_j}$, for $w = n$. It should be noted that the values are independent of the a's.

Furthermore, it is of interest to observe that upon an application of these principles to the polynomial

(6) $$f(i) = c_0 + c_1 i + c_2 i^2 + \cdots + c_n i^n ,$$

one obtains

(7) $$\sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} f(i) = c_n (-1)^n (n!) .$$

University of Illinois (Professor Emeritus)

A RECTIFIED EQUALITY

C. W. Trigg

The January 17 item in R. M. Lucey's *A Problem a Day* (Penguin Books, 1952) requires that the equation

$$2 \; 9 \; 6 \; 7 = 1 \; 7$$

"be rectified by the insertion of simple mathematical signs, without altering the position of any of the figures, changing any figure from one side to the other, or adding any other figures." The solution given is $\sqrt{296-7} = 17$.

A simpler solution is $(2)(9) + 6 - 7 = 17$. Some other solutions are:

$$-2 + 9 - 6 + 7 = 1 + 7 , \quad +2 - 9 + 6 + 7 = -1 + 7 ,$$
$$[2 + (9)(6)]/7 = 1 + 7 , \quad -2 + 9 + 6 - 7 = -1 + 7 ,$$
$$(-2 + 9 - 6)(7) = 1(7) , \quad (-2 + 9 - 6)/7 = 1/7 ,$$
$$(2 - 9)(6 - 7) = 1(7) , \quad (-2 - \sqrt{9} + 6)/7 = 1/7 ,$$
$$(-2 + 9)(-6 + 7) = 1(7) ,$$

and the equations which result from obvious changes of signs which will make both sides of the equations above negative.

There is also at least one solution in the d u o d e c i m a l system of

notation: $(2)(9) - 6 + 7 = 17$.

Los Angeles City College

STATEMENT REQUIRED BY THE ACT OF AUGUST 24, 1912, AS AMENDED BY THE ACTS
OF MARCH 3, 1933, JULY 2, 1946 AND JUNE 11, 1960 (74 STAT. 208) SHOWING THE
OWNERSHIP, MANAGEMENT, AND CIRCULATION OF

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1. The names and addresses of the publisher, editor, managing editor, and business managers are:

                                    Name   Address
Publisher _Robert E. Horton, 2225 W. Silverlake Drive, Los Angeles 39, Calif._

Editor _Robert E. Horton_

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2. The owner is: (If owned by a corporation, its name and address must be stated and also immediately thereunder the names and addresses of stockholders owning or holding 1 percent or more of total amount of stock. If not owned by a corporation, the names and addresses of the individual owners must be given. If owned by a partnership or other unincorporated firm, its name and address, as well as that of each individual member, must be given.)

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5. The average number of copies of each issue of this publication sold or distributed, through the mails or otherwise, to paid subscribers during the 12 months preceding the date shown above was: (This information is required by the act of

June 11, 1960 to be included in all statements regardless of frequency of issue.) _2300._

                                    Robert E. Horton
                            (Signature of editor, publisher, business manager, or owner)

Sworn to and subscribed before me this ___29___ day of _Sept_ , 19 60

        [SEAL]   _____

POD Form 3526
(July 1960)

                (My commission expires _____, 19____)

SOME SPECIAL SERIES

B. E. Rhoades

If a series is arithmetic or geometric, it is possible to write the sum in closed form. The purpose of this note is to show that it is possible to write in closed form series of the form $\Sigma u(k)v(k)$, where $u(k)$ is the k-th term of an arithmetic series (A. S.), and $v(k)$ is the k-th term of a geometric series (G. S.).

The following well-known identities will be useful. For $r \neq 1$,

$$(1) \qquad \sum kr^{k-1} \equiv \frac{nr^{n+1} - (n+1)r^n + 1}{(1-r)^2}$$

$$(2) \qquad \sum k^2 r^{k-1} \equiv \frac{-n^2 r^{n+2} + (2n^2 + 2n - 1)r^{n+1} - (n+1)^2 r^n + r + 1}{(1-r)^3} \, .$$

Throughout this paper the summations are understood to be from 1 to n, unless otherwise indicated.

Identity (1) is proved by differentiating the identity

$$\sum_{r=0}^{n} r^k \equiv \frac{1 - r^{n+1}}{1 - r}$$

with respect to r. To prove (2), multiply (1) by r and differentiate with respect to r.

THEOREM 1. *Let $u(k)$ be the k-th term of an A. S. with difference d and let $v(k)$ be the k-th term of a G. S. with ratio $r \neq 1$. Then*

$$(3) \qquad \sum u(k)v(k) = \frac{v(1)}{(1-r)^2}[(dn+a)r^{n+1} - (a + d(n+1))r^n - ar + d + a] \, ,$$

where $a = u(1) - d$.

By hypothesis, $u(k) = (k-1)d + u(1) = kd + a$, and $v(k) = r^{k-1}v(1)$. Hence

$$\sum u(k)v(k) = v(1)\left[d\sum kr^{k-1} + a\sum r^{k-1}\right].$$

Using (1) along with the formula for the sum of the first n terms of a G. S. yields (3).

THEOREM 2. *If $u(k)$ and $v(k)$ are each k-th terms of A. S.'s with differences d and e, then*

$$(4) \qquad \sum u(k)v(k) = \frac{n}{6}[de(2n+1)(n+1) + 3(ae + bd)(n+1) + 6ab] \, ,$$

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where $a = u(1) - d$, $b = v(1) - e$.

To prove (4), note that

$$\sum u(k)v(k) = de\sum k^2 + (ae + bd)\sum k + ab\sum 1 .$$

Substituting the known sums and simplifying produces (4).

If $u(k) = v(k)$, then (4) simplifies to

(5) $$\sum u^2(k) = \frac{n}{6}[\, d^2(2n+1)(n+1) + 6ad(n+1) + 6a^2\,] .$$

THEOREM 3. *Let* $u(k)$, $v(k)$ *be the k-th terms of A.S.'s with differences* d *and* e, *and let* $w(k)$ *be the k-th term of a G.S. with ratio* $r \neq 1$. *Then, with* $a = u(1) - d$, $b = v(1) - e$,

(6) $$\sum u(k)v(k)w(k) = \frac{w(1)}{(1-r)^3}[\,-\{den^2 + (ae+bd)n + ab\}r^{n+2}$$

$$+\{de(2n^2 + 2n - 1) - (ae + bd)(2n+1) + 2ab\}r^{n+1}$$

$$-\{de(n+1)^2 + (ae + bd)(n+1) + ab\}r^n$$

$$+ abr^2 + \{de - (ae + bd) - 2ab\}r + (a+d)(b+e)] .$$

To prove Theorem 3, note that

$$\sum u(k)v(k)w(k) = w(1)\sum(dek^2 + (ae + bd)k + ab)r^{k-1} ,$$

and use (1) and (2) to obtain the form of (6).

If $u(k) = v(k)$, (6) becomes

(7) $$\sum u^2(k)w(k) = \frac{w(1)}{(1-r)^3}[\,-(dn+a)^2r^{n+2} + \{d^2(2n^2 + 2n - 1)$$

$$+ 2ad(2n+1) + 2a^2\}r^{n+1} - \{d(n+1) + a\}^2 r^n$$

$$+ a^2 r^2 + (d^2 - 2ad - 2a^2)r + (a+d)^2] .$$

The above processes can be continued indefinitely, the only limitation being the amount of algebra involved.

For $|r| < 1$, the series (1), (3), (6), and (7) are partial sums of convergent series, and the sums can be evaluated with ease. For example, (3) tends to

$$\frac{v(1)[d + a(1-r)]}{(1-r)^2} .$$

It is true that any series for which the above theorems are applicable can be summed directly by some other technique. The value of these theorems is in the elimination of repetitive calculations.

The following are examples of (3) to (7) in that order:

$$\sum (2k-1)2^k = 2[\,2^n(2n-3)+3\,]\,,$$

$$\sum k(k+1) = \frac{n(n+1)(n+2)}{3}\,,$$

$$\sum (4k-3)^2 = \frac{n}{3}[16n^2-12n-1]\,,$$

$$\sum \frac{k(2k+1)}{2^{k-1}} = -4[(2n^2+19n+14)(1/2)^{n+1}-7]\,,$$

$$\sum \frac{(2k-1)^2}{3^k} = \frac{-(4n^2+8n+7)(1/3)^n+7}{2}\,.$$

Lafayette College
Easton, Pennsylvania

POLAR SPECIES

One can avoid,
Misanthropoid,
The cardioid.

The plotter knows
How graphing grows
The n-leaf rhos.

He who can skate
A figure eight
Can lemniscate.

Marlow Sholander

GETTING SQUARED AWAY IN 1961

C. W. Trigg

1) $1961 = (1^2 + 6^2)(2^2 + 7^2) = 45^2 - 8^2$.

2) $1961 = 5^2 + 44^2 = 19^2 + 40^2$.

3) $1961 = 1^2 + 14^2 + 42^2 = 3^2 + 4^2 + 44^2 = 4^2 + 24^2 + 37^2$
$= 10^2 + 30^2 + 31^2 = 14^2 + 26^2 + 33^2 = 18^2 + 26^2 + 31^2$
$= 19^2 + 24^2 + 32^2$.

In the following selections from the multitudinous representations of 1961 as the sum of squares, the concise notation $(x_1, x_2, \cdots, x_n)^2 \equiv x_1^2 + x_2^2 + \cdots + x_n^2$ is used.

4) $1961 = (2,7,12,42)^2 = (6,10,12,41)^2 = (1,6,18,40)^2 = (4,10,18,39)^2$.

5) $1961 = (4,6,8,9,42)^2 = (1,2,10,16,40)^2 = (2,4,8,14,41)^2 = (6,9,10,12,40)^2$.

6) $1961 = (1,3,5,7,14,41)^2 = (2,4,8,10,16,39)^2 = (4,6,7,8,14,40)^2$.

7) $1961 = (1,2,3,4,5,15,41)^2 = (1,3,5,7,10,16,39)^2 = (2,3,4,6,10,14,40)^2$.

8) $1961 = (1,2,3,4,8,11,15,39)^2 = (3,4,5,6,7,8,9,41)^2$.

9) $1961 = (1,2,4,5,7,8,9,11,40)^2 = (1,2,3,4,6,7,10,15,39)^2$.

10) $1961 = (1,2,3,4,6,7,9,10,12,39)^2 = (1,2,3,4,5,6,7,11,16,38)^2$.

11) $1961 = (1,2,3,4,5,6,7,9,10,14,38)^2 = (1,2,3,4,5,6,7,8,12,13,38)^2$.

12) $1961 = (4,6,8,9,10,11,12,13,16,17,18,19)^2$
$= (5,6,7,8,9,10,11,13,15,17,19,21)^2$.

13) $1961 = (3,5,7,8,9,10,11,12,13,15,17,18,19)^2$
$= (2,4,6,7,9,10,12,13,14,15,16,18,19)^2$.

14) $1961 = (1,2,3,5,6,7,8,9,10,11,14,17,19,25)^2$
$= (3,4,5,6,7,8,9,10,11,12,15,17,19,21)^2$.

15) $1961 = (1,2,3,6,7,8,9,10,11,13,14,15,16,17,19)^2$
$= (1,2,3,5,6,7,8,9,10,11,14,15,17,19,20)^2$.

16) $1961 = (1,3,4,5,6,7,8,9,10,11,13,14,15,16,17,18)^2$
$= (1,2,3,5,6,7,8,9,10,11,12,14,15,16,17,19)^2$.

17) Square numbers which may be formed from the digits 1, 9, 6, 1 are 16, 169, 196, and 961.

Los Angeles City College

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

A NOTE ON MATHEMATICS AND PHILATELY

C. F. Pinzka

In connection with Maxey Brooke's article, "Mathematics and Philate-
ly," it is worth noting two articles listing stamps which honor mathemati-
cians. These are:

Carl B. Boyer, "Mathematicians and Philately," *Scripta Mathematica,*
vol. 15, pp. 105-114, June, 1949;

H. D. Larsen, "Mathematics and Philately," *American Mathematical
Monthly,* vol. 60, pp. 141-143, February, 1953.

Gauss was honored in 1955 on Germany's 10 pf deep green (Scott 725).
Pascal appears on France's 1.20 fr + 2.80 fr black of 1944(B 181). Henri
Poincaré appears on France's 18 fr + 5 fr dark brown of 1952(B 270).

University of Cincinnati

BOOK REVIEWS

Special Functions. By Earl D. Rainville. Macmillan, New York, 1960, vi +
365 pages. $11.75.

This book contains a compilation of definitions and some properties
of about fifty particular functions, distributed under the following chapter
headings: 1. Infinite products, 2. The Gamma and Beta functions, 3. Asymp-
totic series, 4. The hypergeometric function, 5. Generalized hypergeometric
functions, 6. Bessel functions, 7. The confluent hypergeometric functions,
8. Generating functions, 9. Orthogonal polynomials, 10. Legendre poly-
nomials, 11. Hermite polynomials, 12. Laguerre polynomials, 13. The Scheffer
classification and related topics, 14. Pure recurrence relations, 15. Sym-
bolic relations, 16. Jacobi polynomials, 17. Ultraspherical and Gegenbauer
polynomials, 18. Other polynomial sets, 19. Elliptic functions, 20. Theta
functions and 21. Jacobian elliptic functions.

Many of the special functions are introduced as special hypergeometric
or generalized hypergeometric functions, e. g., Laguerre polynomials and
Jacobi polynomials or by means of a generating function, e. g., the Legendre

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polynomials are *defined* by

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n \ .$$

Each chapter is followed by a set of manipulative type exercises and there is a bibliography at the end of the book.

As an illustration, we shall indicate the treatment of the Gamma and Beta functions in Chapter 2 (25 pages). "The Euler or Mascheroni constant γ" is defined as $\lim_{n \to \infty} [H_n - \text{Log} \, n]$ in which "as usual" $H_n = \sum_{k=1}^{n} \frac{1}{k}$. The existence of the limit is established with the approximation $0 \leq \gamma < 1$. It is stated that "actually $\gamma = 0.5772$ approximately". The function $\Gamma(z)$ is defined by the Weierstrass product and is connected with the Euler integral. The Beta function is defined as an integral and connected with Γ. The formula $\Gamma(z)\Gamma(1-z) = \pi/(\sin \pi z)$ and two other functional equations are established. The approximation $\log \Gamma(z) = (z - \frac{1}{2}) \, \text{Log} \, z - z + \frac{1}{2} \log 2\pi + 0(1)$ is obtained by introducing only the first case of the Euler-Maclaurin sum formula. (The Bernoulli polynomials are given one page in Chapter 18, introduced by the statement: "Much good has come from the study of $B_n(x)$ defined by

$$\frac{te^{xt}}{(e^t - 1)} = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!} \ ,$$

particularly in the Theory of Numbers". No mention is made of their connection with the Euler-Maclaurin sum formula.). Sample exercises are: "Show that $\Gamma'(\frac{1}{2}) = -(\gamma + 2 \, \text{Log} \, 2)\sqrt{\pi}$"; and "Use Euler's integral form $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$ to show that $\Gamma(z+1) = z \, \Gamma(z)$".

This book probably furnishes approximately the "bag of tools" which the usual engineering faculty believes desirable for their students to "cover" in some course.

H. S. Wall

Exposé Moderne des Mathématiques Élementaires. By Lucienne Félix. Dumod, Paris, 1960, 421 pages.

This book is an excellent modern exposition of the structure of Mathematics based on the axiomatic method. The aim of the author is to show the unity of Mathematics — the logical unity in Arithmetic, Algebra, Geometry, and Analysis — through a logical exposition of the concepts, methods,

and the symbols of Modern Mathematics. She succeeds admirably in her attempt to show the unity of Mathematics from an abstract point of view. In this attempt the author constantly uses, on an elementary level, general algebraic structures – groups, rings, fields, vector spaces, and topological structures – to show this unity in the fabric of Mathematics.

The book is divided into four parts.

Part I is about Fundamental Mathematical Structures in which the author explains the following topics on an axiomatic basis : Number System, Vector Spaces, Groups, Introduction to Metric Geometry, and finally Boolean Algebra and a brief introduction to Probability.

Part II is concerned with the Theory of Numbers, Algebraic expressions, and the solution of Algebraic equations.

Part III deals with Analysis. In this section the author discusses Graphs, limits, derivatives, and Complex numbers on an axiomatic basis.

Part IV is about Geometry including Affine Geometry, Fundamental ideas of Projective Geometry, Metrical Geometry, Introduction to Non-Euclidean Geometry, and Conic Sections.

Students majoring in Mathematics, and prospective teachers of Mathematics in High Schools, will find this book very useful. The text is in French.

Souren Babikian

BOOKS RECEIVED FOR REVIEW

Introduction to Symbolic Logic. By A. H. Basson and D. J. O'Connor. The Free Press, Glencoe, Illinois, 1960, viii + 175 pages. $3.00.

Formal Logic of Mathematics. By P. H. Nidditch. The Free Press, Glencoe, Illinois, 1960, vii + 188 pages. $3.00.

Logic of Science and Mathematics. By P. H. Nidditch. The Free Press, Glencoe, Illinois, 1960, vii + 371 pages. $4.00.

Foundations of Modern Analysis. By J. Dieudonné. Academic Press, New York and London, 1960, xiv + 361 pages.

The Foundations of Arithmetic. By Gottlob Frege, translated by J. L. Austin. Harper and Brothers, New York, 1960, xxiii + 119 pages. $1.25.

Matrices and Linear Transformations. By Daniel T. Finkbemer. H. W. Freeman and Company, San Francisco, 1960, vii + 248. $6.50.

National Council of Teachers of Mathematics. Instruction in Arithmetic. Twenty-fifth Yearbook. Washington, D. C., 1960, viii + 366. $3.50.

Automatic Data Processing Systems. By Robert H. Gregory and Richard L. Van Horn. Wadsworth Publishing Company, San Francisco, 1960, xii + 705.

Frontiers of Numerical Mathematics. Edited by Rudolph E. Langer. University of Wisconsin Press, Madison, 1960, xi + 132. $3.50.

Boundary Problems in Differential Equations. Edited by Rudolph E. Langer. University of Wisconsin Press, Madison, 1960, x + 323. $4.00.

Differential Equations. By Tomlinson Fort. Holt, Rinehart and Winston, Inc., New York, 1960, viii + 184. $4.75.

TO A CIRCLE

Oh, thou of equidistant bounds

In each and every octant,

Thy symmetry is well-renowned

Yet mine is mood remonstrant.

No eccentricity redounds

To make thy warmth demonstrant.

Thou art, alas, a bit too round,

Thy curvature too constant.

Marlow Sholander

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

PROPOSALS

432. *Proposed by Lee Tih-Ming, Taipei, Taiwan.*

A point O interior to triangle ABC is joined to the vertices. From O perpendiculars OX, OY, OZ are dropped to the sides BC, CA, AB, respectively. AO and YZ intersect in D, BO and ZX in E, and CO and XY in F. Show that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{ZD}{DY} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ}.$$

433. *Proposed by C. W. Trigg, Los Angeles City College.*

A stone was thrown downward from a roof, at the level of the roof, at an angle of 30° with the horizontal. It passed the upper corner of a rectangular window at 45° with the horizontal and the opposite lower corner at an angle of 60° with the horizontal. If the path of the stone was in a plane parallel to the wall and the window was 6.0 ft. high, find

    a) the width of the window;

    b) the height of the roof above the window;

    c) the speed with which the stone was thrown.

Consider the angles to be given to two-figure accuracy.

434. *Proposed by B. L. Schwartz, Technical Operations, Inc., Honolulu, Hawaii.*

The common fractions 19/95, 26/65, 16/64, and 49/98 can all be reduced to lower terms by "cancelling" the common digits in the numerator and denominator. These are the only proper fractions with two-digit denominators with this property. Characterize the proper fractions with denominators less than 1000 which yield to the same incorrect method.

435. *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Determine the largest and the smallest equilateral triangles that can be inscribed in an ellipse.

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436. *Proposed by Souren Babikian, Los Angeles City College.*
If

$$\tan(\phi + \theta) = \frac{3\tan\theta + \tan^3\theta}{1 + 3\tan^2\theta},$$

show that one value of ϕ is the series

$$\frac{1}{1\cdot 3}\sin 4\theta + \frac{1}{2\cdot 3^2}\sin 8\theta + \frac{1}{3\cdot 3^3}\sin 12\theta + \cdots.$$

437. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*
Prove or disprove the statement: The number of odd coefficients in the binomial expansion of $(a+b)^{[n]}$ is a power of 2, the exponent $[n]$ being the number of 1's appearing in the expression of n in the binary number system.

438. *Proposed by Leon Bankoff, Los Angeles, California.*
Let D be the apex of the equilateral triangle constructed externally on side BC of a triangle ABC, and let Q be the apex of the equilateral triangle constructed on AD, with Q and B on opposite sides of AD. Then let E be the apex of the equilateral triangle on the base CQ, with D and E on opposite sides of CQ. Finally, let F be the apex of the equilateral triangle on the base DE, with Q and F on opposite sides of DE. Show that triangle BAF is equilateral.

SOLUTIONS

Late Solutions

404, 406, 407, 408, 409, 410. *Josef Andersson, Vaxholm, Sweden.*

Erratum

In Problem 430, page 110, Volume 34, Number 2 in the first line, the problem should read, "At a point P on the latus rectum AB of a parabola...".

The 1960 Problemist

411. [May 1960] *Proposed by C.W. Trigg, Los Angeles City College.*
Each letter in this cryptarithm uniquely represents a digit. Reconstruct the factorization.

$$
\begin{array}{r|rrrr}
E & R & T & M & P \\
\hline
 & L & B & T & O \\
 & \cdot\ I & T & S & \\
\hline
 & & R & L &
\end{array}
$$

[Dedicated to RTMP 0123456789]

Solution by C. F. Pinzka, University of Cincinnati.

There are 22 ways of satisfying the relation $(I)(RL) = TS$ if we note the obvious restrictions $I \neq 1$, 5, 9, $R \leq 4$, and L, $S \neq 0$, 1, 5. The condition $(L)(TS) = BTO$ narrows the number of possible solutions to 6. The condition $(E)(BTO) = RTMP$ gives the unique solution

$$
\begin{array}{r|rrr}
5 & 1 & 9 & 6 & 0 \\
& 4 & 3 & 9 & 2 \\
& & 7 & 9 & 8 \\
& & & 1 & 4 \\
\end{array}
$$

[Dedicated to 1960 PROBLEMIST]

Also solved by: Josef Andersson, Vaxholm, Sweden; Maxey Brooke, Sweeny, Texas; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Sidney Kravitz, Dover, New Jersey; Hazel S. Wilson, Jacksonville University, Florida; Dale Woods, Oklahoma State University; and the proposer.

Projective Correspondence

412. [May 1960] *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point on the circumcircle of triangle ABC. Rays from B and C through P meet CA and AB at points E and F respectively. Considering the segments involved as directed quantities, show that

$$
\frac{b^2}{a^2} \cdot \frac{BF}{FA} + \frac{c^2}{a^2} \cdot \frac{CE}{EA} = -1 \,,
$$

where a, b, and c are the sides opposite the vertices A, B, and C of triangle ABC.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.

Having a projective correspondence between the points E and F, we have, letting $e = CE/EA$, $f = BF/FA$, the bilinear relation

$$
A \cdot ef + B \cdot e + C \cdot f + D = 0
$$

where A, B, C, D are constants. To find the values of these coefficients we let P coincide with the points A, B, C successively. If $P = A$, e and f are infinite and $A = 0$. If $P = B$, then BE is an exsymedian; and we have $e = -a^2/c^2$, $f = 0$ and hence

$$
-\frac{B \cdot a^2}{c^2} + D = 0 \quad \text{or} \quad B = \frac{c^2}{a^2} D
$$

and similarly

$$
-\frac{C \cdot a^2}{b^2} + D = 0 \quad \text{or} \quad C = \frac{b^2}{a^2} D \,.
$$

Substitution gives the required result.

Also solved by Josef Andersson, Vaxholm, Sweden; Leon Bankoff, Los Angeles, California; A. F. Hordam, University of New England, Armidale, NSW, Australia; and the proposer.

Fibonacci Determinants

413. [May 1960] *Proposed by the late Victor Thébault, Tennie, Sarthe, France.*

If a, b, c, d, e, and f are consecutive terms of the Fibonacci Series; 1, 1, 2, 3, 5, 8, $\cdots$, prove that

$$\begin{vmatrix} a & b & x-a-b \\ b & c & x-b-c \\ c & d & x-c-d \end{vmatrix} \times \begin{vmatrix} b & c & x-b-c \\ c & d & x-c-d \\ d & e & x-d-e \end{vmatrix} = \begin{vmatrix} c & d & x^2-c-d \\ d & e & x^2-d-e \\ e & f & x^2-e-f \end{vmatrix}.$$

Solution by A. F. Hordam, University of New England, Armidale, NSW, Australia.

If F_1, F_2, F_3, $\cdots$ are successive terms of the Fibonacci sequence, then

$$(1) \qquad\qquad F_n + F_{n+1} = F_{n+2}$$

$$(2) \qquad\qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

In each determinant, add columns 1 and 2 to column 3, then add row 2 to row 1, use (1), and subtract row 3 from row 1. Then L.H.S. becomes

$$\begin{vmatrix} 0 & 0 & x \\ b & c & x \\ c & d & x \end{vmatrix}\begin{vmatrix} 0 & 0 & x \\ c & d & x \\ d & e & x \end{vmatrix} = x(bd-c^2)\cdot x(ce-d^2)$$

$$= x^2 \cdot (-1)^n(-1)^{n+1} \quad \text{by (2)}$$

$$= -x^2,$$

while R.H.S. yields

$$\begin{vmatrix} 0 & 0 & x \\ d & e & x \\ e & f & x \end{vmatrix} = x^2(df-e^2)$$

$$= x^2(-1)^{n+2}$$

$$= (-1)^n x^2.$$

Therefore,

$$\begin{vmatrix} a & b & x-a-b \\ b & c & x-b-c \\ c & d & x-c-d \end{vmatrix}\begin{vmatrix} b & c & x-b-c \\ c & d & x-c-d \\ d & e & x-d-e \end{vmatrix} = \pm \begin{vmatrix} c & d & x^2-c-d \\ d & e & x^2-d-e \\ e & f & x^2-e-f \end{vmatrix}$$

depending on which term of the Fibonacci sequence we start with. For

instance, if we identify a, b, c, d, e, f with 1, 1, 2, 3, 5, 8 i. e. $a = F_1$, then we have $(-x) \cdot x = +(-x^2)$, whereas if we identify a, b, d, d, e, f with 1, 2, 3, 5, 8, 13 i. e. $a = F_2$, then we have $x \cdot (-x) = -(x^2)$.

Also solved by Josef Andersson, Vaxholm, Sweden; J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University; Allen W. Brunson, Fenn College, Cleveland, Ohio; L. Carlitz, Duke University; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; James C. Ferguson, Lynnwood, Washington; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; S. Lajos, University of Belgrade, Yugoslavia; Wahin Ng, San Francisco, California; C. F. Pinzka, University of Cincinnati; William Squire, Southwest Research Institute, San Antonio, Texas; C. W. Trigg, Los Angeles City College; Harvey Walden, Rensselaer Polytechnic Institute, New York; and the proposer.

A Special Slide Rule

414. [May 1960] *Proposed by Sidney Kravitz, Dover, New Jersey.*

If the calculation $f(x) + g(y) = h(z)$ is to be performed, it is possible to construct a slide rule for the purpose. Show that the following formulas lend themselves to calculation on a special slide rule:

(1) $$z = x + xy + y$$

(2) $$z = \frac{x+y}{1+xy} \, .$$

Solution by William Squire, Southwest Research Institute, San Antonio, Texas.

The condition $h(z) = f(x) + f(y)$ is evidently equivalent to

$$\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = f'(x) : f'(y) \, .$$

For $z = x + xy + y$ this relation gives

$$\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = (1+y) : (1+x)$$

and we have

$$\ln (1+z) = \ln (1+x) + \ln (1+y) \, .$$

For $z = \dfrac{x+y}{1+xy}$ we have

$$\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = (1-y^2) : (1-x^2)$$

and we find

$$\ln \frac{1+z}{1-z} = \ln \frac{1+x}{1-x} + \ln \frac{1+y}{1-y} \, .$$

Also solved by Josef Andersson, Vaxholm, Sweden; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; C. F. Pinzka, University of Cincinnati; and the proposer.

A Trigonometric Sum

415. [May 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*
Prove

$$\sum_{p=0}^{n} \binom{n}{p} \cos(p)x \sin(n-p)x = 2^{n-1} \sin nx .$$

Solution by Josef Andersson, Vaxholm, Sweden. (Translated and paraphrased by the editor.)
Making use of the formulas

$$\sum_{p=0}^{n} \binom{n}{p} = 2^n \quad \text{and} \quad \binom{n}{n-p} = \binom{n}{p} ,$$

the original sum can be written

$$\frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} \sin nx + \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} \sin(n-2p)x = 2^{n-1} \sin nx + \frac{s}{2} .$$

It remains to be proven that $s = 0$. Now from the substitution $p = n - p'$ it follows that

$$s = \sum_{p'=n}^{0} \binom{n}{n-p'} \sin(2p'-n) = -s .$$

Therefore $s = 0$.

Also solved by J. L. Brown, Ordance Research Laboratory, Pennsylvania State University; L. Carlitz, Duke University; James C. Ferguson, Lynnwood, Washington; A. F. Hordam, University of New England, Armidale, NSW, Australia; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; William Squire, Southwest Research Institute, San Antonio, Texas; Chih-Yi Wang, University of Minnesota, and the proposer.

An Improper Integral

416. [May 1960] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.* Evaluate

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n} \int_{0}^{\infty} \frac{e^{-x} x^n}{(n-k)! \, n^k} \, dx .$$

Solution by Chih-Yi Wang, University of Minnesota.
By definition of the Gamma function the finite summation is equal to

$$\sum_{k=1}^{n} \frac{n!\,k}{(n-k)!\,n^{k+1}} = n! \sum_{r=0}^{n-1} \frac{n-r}{r!} (\frac{1}{n})^{n-r+1}$$

$$= n! \left\{ \sum_{r=0}^{n-1} \frac{1}{r!} (\frac{1}{n})^{n-r} - \sum_{r=1}^{n-1} \frac{1}{(r-1)!} (\frac{1}{n})^{n-r+1} \right\}$$

$$= n! \left\{ \sum_{r=0}^{n-1} \frac{1}{r!} (\frac{1}{n})^{n-r} - \sum_{s=0}^{n-2} \frac{1}{s!} (\frac{1}{n})^{n-s} \right\}$$

$$= \frac{n!}{(n-1)!} (\frac{1}{n}) = 1 ,$$

which is independent of n.

Also solved by Josef Andersson, Vaxholm, Sweden; William C. Teach-out, Jr., Memphis, Tennessee; and the proposer. One incorrect solution was received.

The Busy Messenger

417. [May 1960] *Proposed by Monte Dernham, San Francisco, California.*
The proverbial messenger, whose favorite pastime is to ride from the rear of a marching column to the front and back to the rear, has been at it again. On a recent occasion, a column x miles long advanced y miles while the messenger was thus motorcycling. On a similar occasion, when he happened to be on horseback, a column two-thirds as long advanced three times as far. After figuring a bit, the messenger was surprised to discover that on each occasion he had traveled exactly y^2 miles. Assuming all speeds were uniform, find x and y.

Solution by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.
Let the rates of the column, the messenger on motorcycle, and the messenger on horseback be r_c, r_m, and r_h, respectively. Let y_m and y_h be the distances traversed by the column during the time the messenger was advancing to the head of the column by motorcycle and horseback, respectively. The following relations hold

$$(1) \qquad \frac{x+y_m}{y_m} = \frac{r_m}{r_c} \;;\quad \frac{x-y+y_m}{r_m} = \frac{y-y_m}{r_c} = \frac{x}{r_c+r_m}$$

$$(2) \qquad \frac{2x/3+y_h}{y_h} = \frac{r_h}{r_c} \;;\quad \frac{2x/3+y_h-3y}{r_h} = \frac{3y-y_h}{r_c} = \frac{2x/3}{r_c+r_h}$$

(3) $$2(x - y + y_m) + y = y^2$$

(4) $$2(2x/3 + y_h - 3y) + 3y = y^2 .$$

Eliminating y_m, r_m, and r_c from (1) and (3) gives $2x + 1 = y^2$. Eliminating y_h, r_m, and r_c from (2) and (4) gives $4x + 27 = 3y^2$. Solving gives $x = 12$ miles and $y = 5$ miles.

Also solved by Josef Andersson, Vaxholm, Sweden; K. L. Cappel, Franklin Institute, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; C.M. Sidlo, Framingham, Massachusetts; C.W. Trigg, Los Angeles City College; and the proposer. Three incorrect solutions were received.

Comment on Problem 399

399. [January and September 1960] *Proposed by Nathan Altshiller Court, University of Oklahoma.*

Prove that the feet of the perpendiculars dropped upon the sides of a triangle from their respective Simson poles form a cevian triangle. That is, the lines joining those points to the respectively opposite vertices are concurrent.

Comment by the proposer.

After having given an elegant and concise synthetic proof of proposal 339, Sister M. Stephanie calls attention to another property of the point involved in the question and suggests an analytic proof for it (this Journal, Vol. 34, 1960, p. 53).

It may be shown that the latter proposition is a special case of the following.

Let AMP, BMQ, CMR be three given concurrent cevians of a triangle ABC, P_0, Q_0, R_0, their respective midpoints, and $AM'P'$, $BM'Q'$, $CM'R'$ their respective isotomic cevians (N.A.C., *College Geometry*, sec. ed., p. 161, Art. 335, New York, 1952).

The points P, P' being isotomic, the line P_0A' joining P_0 to the midpoint A' of BC is parallel to the line AP'. On the other hand, A' is the complementary point of A for ABC, hence the line P_0A' is the complementary of AP' for ABC, and therefore the complementary point L of the point M' of AP' lies on the line P_0A'. For analogous reasons the point L lies on each of the lines Q_0B', R_0C', where B', C' are the midpoints of CA, AB, respectively. Consequently:

a. The three lines joining the midpoints of three concurrent cevians of a triangle to the midpoints of the corresponding sides, have a point in common (*G. de Longchamps, Journal de Mathématiques Élémentaires,* 1885, p. 265).

b. That common point is the complementary, for the triangle, of the

isotomic of the common point of the three given cevians (N. A. C., *Mathesis*, Vol. 67, 1957, p. 261).

In the special case when the given cevians are the altitudes of the triangle, the point L coincides with the Lemoine point (*College Geometry*, p. 256, Art. 586).

Comment on Problem 36

36. [March 1949 and January 1950] *Proposed by Julius Sumner Miller, Michigan College of Mining and Technology.*

A gun can put a projectile to a height R/n, R being the radius of the earth. Assuming the variation in gravitational force with altitude, find the area commanded by the gun.

Comment by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.

The solution as published was $A = 4\pi R^2/n^2$. I wish to show that this result is incorrect and that the correct solution is

$$A = 2\pi R^2[1 - (1 - 1/n^2)^{1/2}] .$$

The fault lies in the use of an incorrect formula for the spherical cap. In the last line of the published solution one finds

$$A = 2\pi R^2(1 - \cos 2\beta)$$

which should be

$$A = 2\pi R^2(1 - \cos \beta),$$

where

$$\beta = \pi - \arccos [-(1 - 1/n^2)^{1/2}] .$$

Hence

$$\cos \beta = (1 - 1/n^2)^{1/2} ,$$

and

$$A = 2\pi R^2[1 - (1 - 1/n^2)^{1/2}] , \quad n > 1 .$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 272. If P is a point on side BC of a triangle ABC, we know by Stewart's theorem that $AB^2 \cdot PC + AC^2 \cdot BP = AP^2 \cdot BC$, with equality when P coincides with either B or C. Show that $AB \cdot PC + AC \cdot BP \geqq AP \cdot BC$. [*Submitted by Leon Bankoff*]

Q 273. The product of four consecutive odd integers is 3313036881. Find the integers. [*Submitted by C. W. Trigg*]

Q 274. Find the general solution of the Diophantine equation

$$(x^4 + y^4 + z^4)^2 = 2(x^8 + y^8 + z^8) .$$

[*Submitted by M. S. Klamkin*]

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 41. In the triangle ABC, BD and BE are trisectors of angle B, while CD and CE are trisectors of angle C. E is the point closer to side BC. Prove that angles BDE and EDC are equal. [*Submitted by C. F. Pinzka*]

T 42. Find a five-digit prime number, the sum of whose digits is 21. [*Submitted by C. W. Trigg*]

T 43. Determine integers a and b such that $x^{15} + ax + b = 0$ and $5^{13} - 233x - 144 = 0$ have a common factor. [*Submitted by M. S. Klamkin*]

(*Answers to Quickies and Solutions to Trickies are on page 184.*)

STROBOGRAMMATIC YEARS

J. M. Howell

Have you noted that our new year, 1961, reads the same when rotated in the plane of the paper through 180 degrees? Perhaps we might call such years strobogrammatic, from the Greek, indicating turned writing. Using our present notation for numbers, there have been twenty-three strobogrammatic years since the first year anno Domimi. However, it will be 40 centuries before another strobogrammatic year occurs. Can you list the strobogrammatic years to date? Check your list with the one on page 184.

Los Angeles City College

Klein bottle

(Answers to Quickies and Solutions to Trickies appearing on pages 181-182.)

ANSWERS

A 272. If $ABPC$ is considered a degenerate quadrilateral, the result follows from Ptolemy's theorem, with equality when A, B, P and C are concyclic. That is, when P coincides with B or C.

A 273. Since N does not end in 5, the units digits of the four numbers in order are 7, 9, 1, and 3. Now $N \doteq 33.13 \times 10^8$, so $\sqrt{N} \doteq 5.76 \times 10^4$, and $\sqrt[4]{N} \doteq 2.4 \times 10^2$. Hence the four numbers are 237, 239, 241, and 243.

A 274. The equation can be factored into

$$(x^2 + y^2 + z^2)(x^2 + y^2 - z^2)(y^2 + z^2 - x^2)(z^2 + x^2 - y^2) = 0 \ .$$

Consequently, the general solution is given by the complete solution to an integral right triangle. That is, $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$, and permutations.

SOLUTIONS

S 41. Since E is the incenter of triangle BCD, the desired result follows immediately.

S 42. Don't waste your time. If the sum of the digits is 21, the number is divisible by 3, hence cannot be prime.

S 43. Assume the common factor has the form $x^2 = mx + n$. If $m = n = 1$, then $x^{13} = F_{13}x + F_{12}$ where F_n are the Fibonacci numbers 1, 1, 2, 3, 5, 8, $\cdots$, and $F_{13} = 233$ while $F_{14} = 144$. Consequently, $-a = F_{15} = 610$ and $-b = F_{14} = 377$. Whether or not other solutions exist is a considerably more involved problem.

STROBOGRAMMATIC YEARS

1, 8, 11, 69, 88, 96, 101, 111, 181, 609, 619, 689, 808, 818, 888, 906, 916, 986, 1001, 1111, 1691, 1881, 1961, 6009.

GEOMETRY IS EVERYWHERE

From: *X-Ray Crystallography*. By M. J. Buerger. John Wiley & Sons, Inc., New York, 1942, pp. 150-152.

As the crystal rotates the plane normal and its lattice points sweep out the surface of a cone of half angle ρ (see Fig. 1). The intersection of this

Figure 1.   J. R. Barnwell

conical curface with the surface of the sphere of reflection is the locus of points where the line of lattice points normal to the plane intersects the sphere. ... The projection of these curves on a cylindrical film for the plane-slope coordinates ρ differing by 5° intervals is shown in Fig. 2.
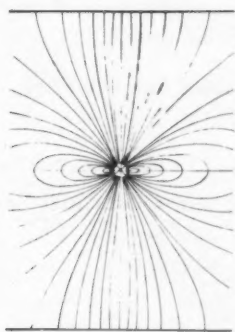
Figure 2.   J. R. Barnwell

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TOO HIGH THE HYPERCUBE

I think I would decline with tact

If asked to meet a tesseract.

I fear I'd play the rube of rubes

Confronted by the cube of cubes

And square factorial of squares

Collected in the cloak he wears.

It's rumored that a cube in time

Can be this supercube sublime,

But in his presence, I confess,

I'd feel like one dimensionless.

Marlow Sholander